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$\mathcal{G}^{+++}$ invariant formulation of gravity and M-theories: exact BPS solutions

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Abstract: We present a tentative formulation of theories of gravity with suitable matter content, including in particular pure gravity in D dimensions, the bosonic effective actions of M-theory and of the bosonic string, in terms of actions invariant under very-extended Kac-Moody algebras $\mathcal{G}^{+++}$. We conjecture that they host additional degrees of freedom not contained in the conventional theories. The actions are constructed in a recursive way from a level expansion for all very-extended algebras $\mathcal{G}^{+++}$. They constitute non-linear realisations on cosets, a priori unrelated to space-time, obtained from a modified Chevalley involution. Exact solutions are found for all $\mathcal{G}^{+++}$. They describe the algebraic properties of BPS extremal branes, Kaluza-Klein waves and Kaluza-Klein monopoles. They illustrate the generalisation to all $\mathcal{G}^{+++}$ invariant theories of the well-known duality properties of string theories by expressing duality as Weyl invariance in $\mathcal{G}^{+++}$. Space-time is expected to be generated dynamically. In the level decomposition of $E_8^{+++} = E_{11}$, one may indeed select an A_{10} representation of generators P_a which appears to engender space-time translations by inducing infinite towers of fields interpretable as field derivatives in space and time.

Keywords: p-branes, String Duality, Space-Time Symmetries, Classical Theories of Gravity.

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Contents

1. Introduction

String theories, and particularly superstrings and their possible unification at the non-perturbative level in an elusive M-theory, are often viewed in the double perspective of a consistent gravity theory and of fundamental interaction unification. Matter degrees of freedom relevant for the unification program are identified with the huge number of degrees of freedom present in the string modes. Consistency with quantum gravity requirements at the perturbative level and impressive theoretical successes, among which the evaluation of the statistical entropy of near extremal black holes [1] is probably the most significant, seem to indicate that the string-M-theory approach contains elements of a consistent theory of gravity and matter.

The project however stumbles when confronting gravity at the non perturbative level. Our point of view is that, although the introduction of the new degrees of freedom in string theories is essential and constitutes a clue of a correct computation of the black hole entropy, some new physical ingredient, which is to some extend foreign to the unification paradigm, is needed to cope with gravity. The present work is an attempt to uncover such ingredient.

We take advantage of two different trends.

The first one is the interpretation of the well-known symmetry of the scalar cosets emerging in the dimensional reduction of gravity, suitably coupled to forms and to dilatons, as a remnant of a much larger symmetry of the full covariant and gauge invariant original theory. Coset symmetries were first found in the dimensional reduction of eleven-dimensional supergravity [2] but appeared also in other theories. They have been the subject of much study, and some classic example are given in [3]. In fact, all simple maximally non-compact Lie group $\mathcal{G}$ could be generated from the reduction down to three dimensions of suitably chosen actions [4]. It was conjectured that these actions, or some unknown extension of them, possess the much larger very-extended Kac-Moody symmetries $\mathcal{G}^{+++}$. $\mathcal{G}^{+++}$ algebras are defined from the Dynkin diagrams obtained from those of $\mathcal{G}$ by adding three nodes [5]. One first adds the affine node, then a second node connected to it by a single line and then similarly a third one connected to the second. These define respectively the affine Kac-Moody algebra $\mathcal{G}^{+}$, the overextended and very extended lorentzian Kac-Moody algebras $\mathcal{G}^{++}$ and $\mathcal{G}^{+++}$. The conjecture of a $\mathcal{G}^{+++}$ symmetry originated from the study of generalisations to gravity and to form field strengths of the considerations which produced the scalar cosets. The E_8 invariance of the dimensional reduction to three dimensions of 11-dimensional supergravity would be enlarged to $E_8^{+++} = E_{11}$, as first proposed in reference [6]. Similarly the effective action of the bosonic string would have the symmetry $D_{24}^{+++} = k_{27}$ [6]. A_D^{+++} was also proposed [7] for pure gravity in D space-time dimensions. It was then shown that some solutions of general relativity and dilatons form representations of the Weyl group of $\mathcal{G}^{+++}$ for *all* actions S, which dimensionally reduced to three dimensions lead to a Lie group $\mathcal{G}$ symmetry, and the extension to $\mathcal{G}^{+++}$ was proposed in general [8]. In a different development, the study of the properties of cosmological solutions in the vicinity of a space-like singularity, known as cosmological billiards [9], revealed an overextended symmetry $\mathcal{G}^{++}$ for all $\mathcal{G}$ [10, 11].

The second one stems from the remarkable attempt to recover, in the case of M-theory, the bosonic equations of motion of supergravity out of the overextended symmetry $E_8^{++} = E_{10}$.[1] An E_8^{++}-invariant lagrangian was proposed [12]. It was built in a recursive way on the coset E_8^{++}/K^{++}, where K^{++} is the subalgebra of E_8^{++} invariant under the Chevalley involution, by a 'level' expansion in terms of the subalgebra A_9. The level of an irreducible representation of A_9 counts the number of times the special root α_{11} in

[1]In the context of dimensional reduction, the appearance of E_{10} in one dimension has been first conjectured by B. Julia [13].

the Dynkin diagram of E_8^{++} (that is the E_8^{+++} diagram of Fig.1 with the node 1 erased) appears in the decomposition of the adjoint representation of E_8^{++} into A_9 representations. The theory was formulated as a perturbation expansion near a space-like singularity and checked up to the third level. The supergravity fields were taken to depend on time only. Space derivatives were supposed to be hidden in higher level objects, together with new degrees of freedom hopefully related to the Hilbert space of superstrings.

The overextended $\mathcal{G}^{++}$ leaves out naturally time as the special coordinate, which was explicitly introduced in the non-linear realisation of E_8^{++}. Here, we will formulate a $\mathcal{G}^{+++}$ invariant theory by putting all space-time coordinates on the same footing by defining a map to $\mathcal{G}^{+++}$ of a world-line *a priori unrelated to space-time*. The latter should then be deduced dynamically. Such an approach to gravity and forms, if successful, would dispose of the need of explicit diffeomorphism invariance or gauge invariance. All such information should be hidden in the *global* $\mathcal{G}^{+++}$ invariance. Although it may seem that global symmetries cannot contain local symmetries, in particular in view of the celebrated Elitzur theorem [14], this need not be the case in view of the infinite number of generators of $\mathcal{G}^{+++}$. We formulate the $\mathcal{G}^{+++}$ invariant theory from a level decomposition[2] with respect to a subalgebra A_{D-1} where D turns out to be the space-time dimension. Our formulation is exploratory and does not pretend to be a final one. No attempt is made to cope with fermionic degrees of freedom and we limit here our investigation to the classical domain. Note however that the very fact that the theory is not formulated in space time and that it rests on a huge symmetry opens new perspectives for the quantisation procedure.

To test the validity of our approach, we derive and discuss exact 'BPS' solutions of the $\mathcal{G}^{+++}$ invariant action and compare them to BPS solutions of maximally oxidised theories discussed in [17]. We obtain in this way the full algebraic structure of the BPS solutions of these conventional actions and put into evidence their group-theoretical significance. This approach does not yield direct information about their space-time behaviour but we find indications on how space-time can be encoded in the $\mathcal{G}^{+++}$ invariant theories. One can indeed select in the level decomposition of $E_8^{+++} = E_{11}$ an A_{10} representation of generators P_a that appears to engender space-time translations. A more detailed analysis of a dictionary translating the content of $\mathcal{G}^{+++}$ into space-time language is differed to a separate publication [18]. Also the relation between our approach and the 'hamiltonian' overextended $\mathcal{G}^{++}$-theories, in which time is explicitly introduced, will be discussed elsewhere [19].

The paper is organised as follows. In section 2, we construct in a recursive way, for any $\mathcal{G}^{+++}$, an action invariant under non-linear transformations of $\mathcal{G}^{+++}$. The action is defined in a reparametrisation invariant way on a world-line, a priori unrelated to space-time, in terms of fields $\phi(\xi)$ where ξ spans the world-line. We use a level decomposition of $\mathcal{G}^{+++}$ with respect to its subalgebra A_{D-1} where the integer D is related to the rank of $\mathcal{G}^{+++}$. The fields $\phi(\xi)$ live in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subalgebra K^{+++} is invariant under a 'temporal involution' which is different from the often used Chevalley involution. The temporal involution preserves the Lorentz algebra $\mathrm{SO}(D-1,1)$. As a consequence,

[2]Level expansions of very-extended algebras in terms of the subalgebra A_{D-1} have been considered in [15, 16].

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the action is Lorentz invariant at each level and all fields, which can be transformed between themselves by general $\mathcal{G}^{+++}$ transformations live on a coset $\mathrm{GL}(D)/\mathrm{SO}(D-1,1)$. This allows the identification of each ξ-field to a field defined at a *fixed* space-time point, independently of ξ. In section 3, we find exact solutions of our invariant action for all $\mathcal{G}^{+++}$. These completely define the algebraic properties of extremal BPS branes, Kaluza-Klein waves and Kaluza-Klein monopoles (Taub-NUT space-times) [17] and the motion in ξ-space is consistent with a motion in the space of solutions. The transformations under the $\mathcal{G}^{+++}$ Weyl group of these solutions put into light the generality of duality transformations which are often considered as characteristic of string theories and supersymmetry [20]–[22]. The only element missing is the space-time dependance of a harmonic function entering the space-time description of the solutions. However, we find indications that space-time properties are included in higher levels. An A_{10} representation, possibly related to space-time translations by inducing infinite towers of fields interpretable as field derivatives in space and time, is exhibited for $E_8^{+++} = E_{11}$ at level seven. In the concluding section 4, we summarise and discuss our results. Detailed computations, in particular of Weyl reflections in all $\mathcal{G}^{+++}$, and illustrations of the step operator transformations as duality transformations in E_8^{+++} are given in the appendix A–C.

2. Non-linear realisation of very extended algebras

2.1 Preliminaries

Theories of gravity in D space-time dimensions coupled to q scalar 'dilatons' and p_I-form field strengths exhibit, upon dimensional reduction to three space-time dimensions, the global symmetry $\mathrm{GL}(D-3) \times \mathrm{U}(1)^q$ of deformations of the compact $(D-3)$ torus and of dilaton field translations. For well chosen D, p_I, q and dilaton couplings to the p_I-forms, the group $\mathrm{GL}(D-3) \times \mathrm{U}(1)^q$ of rank $(D-3+q)$ in enhanced to a simple maximally non-compact Lie group $\mathcal{G}$ of the same rank. The scalar fields of the reduced lagrangian form a non-linear realisation of $\mathcal{G}$. They live on the coset space[3] $\mathcal{G}/\mathcal{H}$ where $\mathcal{H}$ is the maximal compact subgroup of $\mathcal{G}$.

For each group $\mathcal{G}$, we select a maximally oxidised theory,[4] that is a theory that is not the dimensional reduction of a higher dimensional one and has the symmetry $\mathcal{G}$ when reduced to three dimensions [4]. We write the corresponding lagrangian as

$$S = \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \left[R - \frac{1}{2} \sum_{u=1}^{q} (\partial \phi^u)^2 - \frac{1}{2} \sum_I \frac{1}{p_I!} \exp\left(\sum_{u=1}^{q} a_I^u \phi^u \right) F_{p_I}^2 \right] + \text{C.S.}\,, \tag{2.1}$$

where C.S. represents Chern-Simons terms that are required for some groups $\mathcal{G}$. The relation between D, q and $\mathcal{G}$ is summarised in Table I. The actions eq. (2.1) include the theories of pure gravity for $\mathcal{G} = A_{D-3}$, the bosonic sector of the M-theory effective action for $\mathcal{G} = E_8$ and the bosonic string effective action for $\mathcal{G} = D_{24}$.

[3] For a general pedagogical review see [23].

[4] We consider here only theories expressible in lagrangian form.

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q	D	$\mathcal{G}$
0	D	A_{D-3}
1	D	B_{D-2} and D_{D-2}
q	4	C_{q+1}
1	8	E_6
1	9	E_7
0	11	E_8
0	5	G_2
1	6	F_4

Table 1: q and D for each simple Lie group G in the reduced theory.

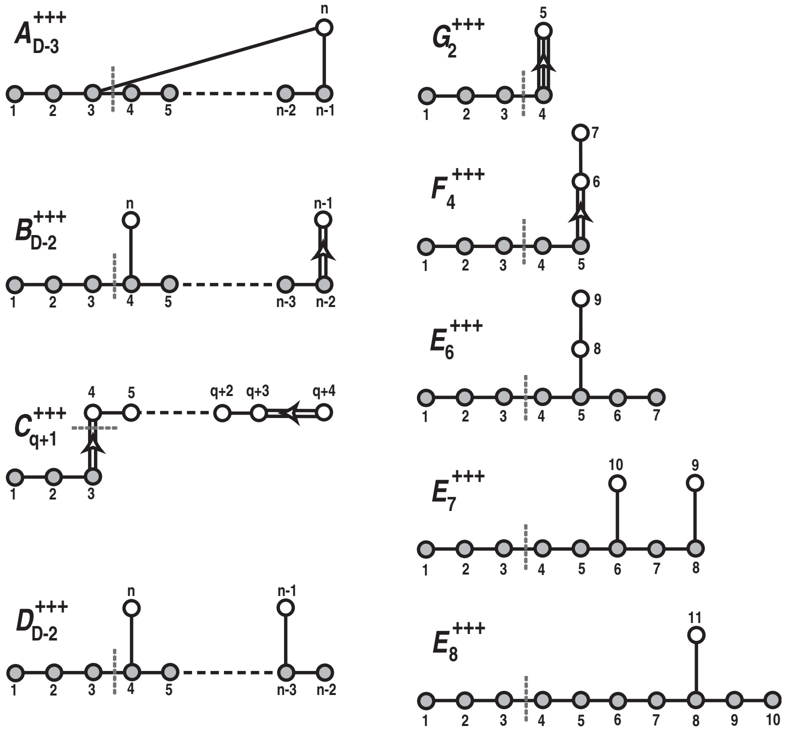

Figure 1: Dynkin diagram of $\mathcal{G}^{+++}$.The nodes of the gravity line are shaded. The Dynkin diagram of $\mathcal{G}$ is that part of the diagram of $\mathcal{G}^{+++}$ which sits on the right of the dashed line. The first three nodes define the Kac-Moody extensions.

We consider the real form of the symmetrisable lorentzian Kac-Moody algebra $\mathcal{G}^{+++}$ defined from the algebra of $\mathcal{G}$ by extending its Dynkin diagram with three nodes in a line [5], the first one being the affine node extending $\mathcal{G}$ to its affine Kac-Moody extension.[5] The Dynkin diagram of $\mathcal{G}^{+++}$ contains the diagram, hereafter referred to as *the gravity line*, of the subalgebra $A_{D-1} = SL(D)$. The algebra of $\mathrm{GL}(D) \times \mathrm{U}(1)^q$, which has the same rank as $\mathcal{G}^{+++}$, is always a subalgebra of $\mathcal{G}^{+++}$. The Dynkin diagrams of all simple $\mathcal{G}^{+++}$ are depicted in figure 1.

[5]Throughout the paper, we use the same notation for groups and Lie algebras.

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We want ultimately $\mathcal{G}^{+++}$ to fully characterise the symmetry of the action S defined by eq. (2.1), or of some more general theory. As an enlargement of the *global* symmetry group $\mathcal{G}$ arising in the dimensional reduction of S, $\mathcal{G}^{+++}$ should also define a *global* symmetry. This poses a dilemma. The metric and the form field strengths in S are genuine space-time fields and S is invariant under the *local* diffeomorphism and gauge groups. How could a global symmetry encompass a local symmetry? The present analysis is an attempt to solve this dilemma by taking advantage of the infinite dimensionality of the algebra $\mathcal{G}^{+++}$. More precisely we shall replace the action S by an action $\mathcal{S}$ explicitly invariant under the global $\mathcal{G}^{+++}$ symmetry. $\mathcal{S}$ will contain an infinite number of objects that are tensors under $SL(D)$. These will comprise a symmetric tensor $g_{\mu\nu}$, scalars ϕ^u and (p_I-1)-form potentials $A_{\mu_1\mu_2\ldots\mu_{p_I-1}}$ which shall be interpreted as the corresponding fields occurring in eq. (2.1) taken at a *fixed* space time point. Their motion in space-time, as well as those of possible additional fields, is expected to take place through an infinite number of field derivatives at this point, encoded in other objects in $\mathcal{S}$.

We recall that the algebra of $\mathcal{G}^{+++}$, whose rank $r = D + q$, is entirely defined by the commutation relations of its Chevalley generators and by the Serre relations. Let E_m, F_m and $H_m,\ m = 1, 2, \ldots r$, be the generators satisfying

$$\begin{aligned} [H_m, H_n] &= 0\,, & [H_m, E_n] &= A_{mn} E_n\,, \\ [H_m, F_n] &= -A_{mn} F_n\,, & [E_m, F_n] &= \delta_{mn} H_m\,, \end{aligned} \tag{2.2}$$

where A_{mn} is the Cartan matrix. The Cartan subalgebra is generated by H_m, while the positive (negative) step operators are the E_m (F_n) and their multi-commutators subject to the Serre relations

$$[E_m, [E_m, \ldots, [E_m, E_n] \ldots]] = 0\,, \qquad [F_m, [F_m, \ldots, [F_m, F_n] \ldots]] = 0\,, \tag{2.3}$$

where the number of E_m (F_m) acting on E_n (F_n) is given by $1 - A_{mn}$.

The generators of the GL(D) subalgebra are taken to be $K^a{}_b$ ($a, b = 1, 2, \ldots, D$) with commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,. \tag{2.4}$$

The cartan generators H_m in the Chevalley basis are linear combinations of the $K^a{}_a$ and of the abelian generators R_u ($u = 1, 2, \ldots, q$). The Cartan generators corresponding to the gravity line are given by $H_m = K^m_{\ m} - K^{m+1}_{\ \ m+1}$ with $m = 1 \ldots D-1$. We shall see how to express the positive and negative step operators E_α and $(E^T_{-\alpha})$ as $SL(D)$ tensors. Here E_α designate the E_m and all the generators obtained from them by multi-commutators, up to a multiplicative normalisation factor. $E^T_{-\alpha}$ designate the generator obtained by substituting in E_α all E_m by F_m and taking the multi-commutators in reverse order. Their normalisation factor is chosen so that the invariant bilinear form in $\mathcal{G}^{+++}$ [24] is normed to

$$\langle E_{\alpha\, s}, E^T_{-\beta\, t}\rangle = \delta_{st}\, \delta_{\alpha\beta}\,, \tag{2.5}$$

where s and t label possible degeneracies of the root α.

The positive (negative) step operators in the A_{D-1} subalgebra are, from eq. (2.4), the $K^a{}_b$ with $b > a$ ($b < a$). They define the level zero step operators of the $\mathcal{G}^{+++}$ adjoint representation. The positive (negative) levels of the adjoint representation of $\mathcal{G}^{+++}$ are

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defined as follows. One takes a set of q non-negative (non-positive) integers, excluding q zeros, where q is the number of simple roots of $\mathcal{G}^{+++}$ not contained in the gravity line. The q integers count the number of times each such root appears in the decomposition of the adjoint representation of $\mathcal{G}^{+++}$ into irreducible representations of A_{D-1}. Positive (negative) levels contain only positive (negative) roots and the number of irreducible representations of A_{D-1} at each level is finite. All step operators may be written as irreducible tensors of the A_{D-1} subalgebra of $\mathcal{G}^{+++}$. Their symmetry properties are fixed by the Young tableaux describing the irreducible representations appearing at a given level. Iterative procedures to compute the step operators at any level can be devised. They build, together with the Cartan generators $K^a{}_a$ $(a = 1, 2, \ldots, D)$ and R_u $(u = 1, 2, \ldots, q)$, the full content of the adjoint representation of $\mathcal{G}^{+++}$.

The commutators of all step operators are generated by the commutators of step operators corresponding to simple roots. At level zero these 'simple step operators' are, from eq. (2.4), the $K^a{}_{a+1}$ $(a = 1, 2, \ldots, D-1)$. Those not contained in the gravity line are components of tensors occurring at low levels. Tensor transformation properties are given by

$$[K^a{}_b, R_{d_1\ldots d_s}{}^{c_1\ldots c_r}] = \delta^{c_1}_b R_{d_1\ldots d_s}{}^{a\ldots c_r} + \ldots + \delta^{c_r}_b R_{d_1\ldots d_s}{}^{c_1\ldots a} - \delta^a_{d_1} R_{b\ldots d_s}{}^{c_1\ldots c_r} - \ldots - \delta^a_{d_s} R_{d_1\ldots b}{}^{c_1\ldots c_r} \,. \quad (2.6)$$

For instance, we see from the E_8^{+++} Dynkin diagram in figure 1, which characterises M-theory, that the only non-gravitational simple step operator occurs at level one and is associated to the root α_{11}. The only A_{10} representation at that level is a third rank antisymmetric tensor R^{abc}. This is of course to be expected as the 'electric' root α_{11} is generated in the dimensional reduction by a 3-form potential in the action eq. (2.1). One has thus

$$[K^a{}_b, R^{efg}] = \delta^e_b R^{afg} + \delta^f_b R^{eag} + \delta^g_b R^{efa} \,, \quad (2.7)$$

and from the Chevalley relations eq. (2.2) one easily identifies the simple step operator at level one as $R^{9\,10\,11}$. In general, when q dilatons are present in the action S, the rank of $\mathcal{G}^{+++}$ is $D + q$. The q abelian generators R_u of its subgroup $\mathrm{GL}(D) \times \mathrm{U}(1)^q$ have non vanishing commutators with the tensor step operators $R^{a_1 a_2 \ldots a_r}$ associated to electric or magnetic simple roots. In dimensional reduction, these arise from r-form potentials where $r = p_I - 1$ for an electric root and $r = D - p_I - 1$ for a magnetic one. We can read off their commutation relation, in the normalisation given for the dilaton in eq. (2.1), namely

$$[R_u, R^{a_1 a_2 \ldots a_r}] = -\varepsilon \frac{a^u_I}{2} R^{a_1 a_2 \ldots a_r} \,, \quad (2.8)$$

where $\varepsilon = +1$ for an electric root and -1 for a magnetic one [8]. Eqs. (2.4), (2.6) and (2.8) allow to express all step operators associated to the simple roots of $\mathcal{G}^{+++}$ and their commutations relations with the generators of the Cartan subalgebra as A_{D-1} tensors. These are listed in appendix A for all simple algebra $\mathcal{G}^{+++}$. By multiple commutators and reduction at any level into suitably normed irreducible A_{D-1} tensors, one may in principle, recursively, list the positive step operators at any level.

To switch from positive $K^a{}_b\,(b > a)$ step operators to negative ones it suffices to interchange upper and lower indices. Writing the negative of $R_{d_1\ldots d_s}{}^{c_1\ldots c_r}$ as $R^{d_1\ldots d_s}{}_{c_1\ldots c_r}$, one verifies that the correct tensor transformations of the negative step operators follows from those of the positive ones by writing the commutator eq. (2.6) in reverse order, namely $[R^{d_1\ldots d_s}{}_{c_1\ldots c_r}, K^b{}_a]$. One may then form all negative step operators by writing the multi-commutators of the simple roots defining the positive step operators, interchanging in each simple step operator upper and lower indices, and taking the multi-commutators in reverse order. In this way the bilinear form eq. (2.5) becomes

$$\langle R_{b_1\ldots b_s}{}^{a_1\ldots a_r}\, R^{c_1\ldots c_s}{}_{d_1\ldots d_r}\rangle = \delta^{c_1}_{b_1}\cdots\delta^{c_s}_{b_s}\delta^{a_1}_{d_1}\cdots\delta^{a_r}_{d_r}\,. \tag{2.9}$$

2.2 The temporal involution and the coset space $\mathcal{G}^{+++}/K^{+++}$

The metric $g_{\mu\nu}$ at a fixed space time point parametrises the coset $\mathrm{GL}(D)/\,\mathrm{SO}(D-1,1)$. To construct a $\mathcal{G}^{+++}$ invariant action $\mathcal{S}$ containing such a tensor, we shall build a non linear realisation of $\mathcal{G}^{+++}$ in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subgroup K^{+++} contains the Lorentz group $\mathrm{SO}(D-1,1)$. We use a recursive construction based on the level decomposition of $\mathcal{G}^{+++}$. As at each level the $\mathrm{SO}(D-1,1)$ invariance must be realised for a finite number of generators, we cannot use the Chevalley involution to build the coset $\mathcal{G}^{+++}/K^{+++}$. Rather we shall use a 'temporal' involution from which the required non-compact generators of K^{+++} can be selected.

The 'temporal' involution is defined in the following way. For the generators of the Cartan subalgebra we take, as in the Cartan involution,

$$K^a{}_a \mapsto -K^a{}_a \qquad R_u \mapsto -R_u\,, \tag{2.10}$$

or equivalently in the Chevalley basis

$$H_m \mapsto -H_m\,. \tag{2.11}$$

The simple step operators are mapped according to

$$E_m \mapsto -\epsilon_m F_m \qquad F_m \mapsto -\epsilon_m F_m\,. \tag{2.12}$$

Here E_m is expressed as a A_{D-1} tensor and F_m as the tensor with upper and lower indices interchanged. ϵ_m is defined as +1 if the number of '1' indices (that is the number of time indices) is even and (-1) otherwise. It is straightforward to verify that the Chevalley presentation eq. (2.2) is preserved under the map given in eqs. (2.11) and (2.12).

Any positive step operator $E_{\alpha\,s}$ is expressed (up to a normalisation constant) as

$$E_{\alpha\,s} = [E_q, [E_{q-1}, \ldots, [E_2, E_1]\ldots]]\,. \tag{2.13}$$

Under the map eq. (2.12) one gets

$$\begin{aligned} E_{\alpha\,s} &\mapsto (-1)^q\,\epsilon_1\epsilon_2\ldots\epsilon_{q-1}\epsilon_q\,[F_q,[F_{q-1},\ldots,[F_2,F_1]\ldots]] = \\ &= (-1)\,\epsilon_1\epsilon_2\ldots\epsilon_{q-1}\epsilon_q\,[[\ldots[F_1,F_2],\ldots,F_{q-1}],F_q] \\ &= -\epsilon_\alpha\,E^T_{-\alpha\,s}\,. \end{aligned} \tag{2.14}$$

As seen from the previous discussion, the negative step operator $E^T_{-\alpha\,s}$ is obtained by interchanging upper and lower indices in $E_{\alpha\,s}$. The factor ϵ_α is ±1, according to the parity of the number of time indices occurring in $E_{\alpha\,s}$ (or in $E^T_{-\alpha}$). Clearly all commutation relations in $\mathcal{G}^{+++}$ are preserved under the mapping eqs. (2.11) and (2.12), and so is its bilinear form, as seen from eq. (2.5) and (2.9). This mapping constitutes an involution that we label the temporal involution. We define the subgroup K^{+++} of $\mathcal{G}^{+++}$ as the subgroup invariant under this involution. Its generators are

$$E_{\alpha\, s} - \epsilon_\alpha\, E^T_{-\alpha\, s}\,. \tag{2.15}$$

K^{+++} contains the Lorentz group SO$(D-1,1)$ and all generators with $\epsilon_\alpha = -1$ are non-compact.[6]

2.3 Non-linear realisation of $\mathcal{G}^{+++}$ in $\mathcal{G}^{+++}/K^{+++}$

We will follow a similar line of thought as the one developed in reference [12] in the context of E_8^{++}. Consider a group element $\mathcal{V}$ built out of Cartan and positive step operators in $\mathcal{G}^{+++}$. It takes the form

$$\mathcal{V} = \exp\left(\sum_{a\geq b} h_b{}^a K^b{}_a - \sum_{u=1}^{q} \phi^u R_u\right) \exp\left(\sum \frac{1}{r!s!} A_{b_1\ldots b_s}{}^{a_1\ldots a_r} R^{b_1\ldots b_s}{}_{a_1\ldots a_r} + \cdots\right). \tag{2.16}$$

We have written it so that the first exponential contains only level zero operators (i.e the Cartan and the level zero positive step operators) and the second one contains the positive step operators of level strictly greater than zero. The tensors $h_b{}^a, \phi^u, A_{b_1\ldots b_s}{}^{a_1\ldots a_r}$, bear a priori no relation with the metric, the dilaton and the potentials of the p_I form field strengths F_{p_I} entering the action eq. (2.1). However we shall see that a dictionary can be established relating the tensors which appear at low levels with the fields occurring in eq. (2.1) at a *fixed* space-time point. For higher levels the dictionary between group parameters and space-time fields should arise, as discussed in section 3, from the analysis of the dynamics encoded in the $\mathcal{G}^{+++}$ invariant action $\mathcal{S}$ below.

A differential motion in the coset $\mathcal{G}^{+++}/K^{+++}$ can be constructed from eq. (2.16). Define

$$dv = d\mathcal{V}\mathcal{V}^{-1} \qquad d\tilde{v} = \widetilde{\mathcal{V}}^{-1} d\widetilde{\mathcal{V}} \qquad ; \qquad dv_{sym} = \frac{1}{2}(dv + d\tilde{v})\,. \tag{2.17}$$

Here $\widetilde{\mathcal{V}}$ is obtained from $\mathcal{V}$ by the map obtained by flipping the sign in the RHS of eqs. (2.11) and (2.14), namely $H_m \mapsto +H_m$ and $E_{\alpha\, s} \mapsto +\epsilon_\alpha\, E^T_{-\alpha\, s}$. As dv and $d\tilde{v}$ are differentials in the Lie algebra, dv_{sym} contains only the Cartan generators and the combinations of step operators $E_{\alpha\, s} + \epsilon_\alpha\, E^T_{-\alpha\, s}$. Hence it defines a differential motion in the coset $\mathcal{G}^{+++}/K^{+++}$. This linear combination of step operators takes the form $R_{d_1\ldots d_s}{}^{c_1\ldots c_r} + \epsilon_{c_1}\ldots\epsilon_{c_r}\epsilon_{d_1}\ldots\epsilon_{d_s} R^{d_1\ldots d_s}{}_{c_1\ldots c_r}$ where $\epsilon_a = -1$ if $a = 1$ and $\epsilon_a = +1$ otherwise.

[6] The occurrence of a non-compact K^{+++} invariant under the temporal involution is the infinite dimensional counterpart of what is happening in dimensional reduction when one compactifies the time. Indeed, for instance, 11 dimensional supergravity compactified down to three spatial dimensions gives a coset G/H with $G = E_8$ and $H = SO^*(16)$ with $SO^*(16)$ being a non-compact form of SO(16) with maximal compact subgroup U(8) [25].

To construct the action $\mathcal{S}$ we wish to map a manifold $\mathcal{M}$ into $\mathcal{G}^{+++}$. We do not want to take for $\mathcal{M}$ a space-time manifold, as this might require the explicit introduction of local symmetries which we hope to be hidden in the infinite algebra of $\mathcal{G}^{+++}$. We shall take for $\mathcal{M}$ a one-dimensional world-line in ξ-space, i.e. $dv_{sym} = v_{sym}(\xi)d\xi$. Here no connection is imposed a priori between ξ-space and space-time.

A reparametrisation invariant action is then

$$\mathcal{S} = \int d\xi \frac{1}{n(\xi)} \langle v_{\rm sym}^2(\xi) \rangle = \frac{1}{4} \int d\xi \frac{1}{n(\xi)} \left\langle \left(\frac{d\mathcal{V}(\xi)}{d\xi} \mathcal{V}(\xi)^{-1} + \widetilde{\mathcal{V}}^{-1} \frac{d\widetilde{\mathcal{V}}}{d\xi} \right)^2 \right\rangle , \qquad (2.18)$$

where $\mathcal{V}(\xi)$ are the group parameters appearing in eq. (2.16) that are now fields dependent on the variable ξ, and $n(\xi)$ is an arbitrary lapse function ensuring reparametrisation invariance on the world-line. The 'trace' $\langle \ \rangle$ means the invariant bilinear form on $\mathcal{G}^{+++}$ [24] which can in principle be computed in the recursive approach. It ensures the invariance of the non-linear action $\mathcal{S}$ defined on the coset space $\mathcal{G}^{+++}/K^{+++}$ under global $\mathcal{G}^{+++}$ transformations.

We now compute the level zero of the action eq. (2.18), that is the terms generated by $K^b{}_a$ $(a \geq b)$ and R_u in $\mathcal{V}$. Writing $dv = v(\xi)d\xi$, one obtains from eqs. (2.16) and (2.17) the contribution of the level zero to $v_{\rm sym}(\xi)$,

$$v^0_{\rm sym}(\xi) = -\frac{1}{2} \sum_{a \geq b} \left[e^h \left(\frac{de^{-h}}{d\xi} \right) \right]_b^{\ a} (K^b{}_a + \epsilon_a \epsilon_b K^a{}_b) - \sum_{u=1}^{q} \frac{d\phi^u}{d\xi} R_u \, , \qquad (2.19)$$

where h is triangular matrix with elements $h_b{}^a$. We now evaluate $\langle (v^0_{\rm sym})^2 \rangle$. The invariant form in $\mathcal{G}^{+++}$ for the Cartan generators is given by [8]

$$\langle K^a_a K^b_b \rangle = G_{ab} \qquad \text{with} \qquad G = I_D - \frac{1}{2} \Xi_D \, , \qquad \langle R_u R_v \rangle = \frac{1}{2} \delta_{uv} \, , \qquad (2.20)$$

where Ξ_D is a D-dimensional matrix with all entries equal to unity. For the step operators we have

$$\langle K^b{}_a K^d{}_c \rangle = \delta^b_c \delta^d_a \qquad a > b \, . \qquad (2.21)$$

The ϵ-symbols defining the temporal involution allow the raising or lowering of the a, b indices of the ξ-fields multiplying the negative step operator in $\langle v^0_{\rm sym}(\xi) v^0_{\rm sym}(\xi) \rangle$ with the minkowskian metric η_{ab} . This ensures that this expression is a Lorentz scalar. We get thus the Lorentz invariant action at level zero, $\mathcal{S}^0$, using eqs. (2.20), (2.21) and the triangular structure of h,

$$\begin{aligned} \mathcal{S}^{(0)} &= \int d\xi \frac{1}{n(\xi)} \langle (v^0_{\rm sym}(\xi))^2 \rangle \, , \\ &= \frac{1}{2} \int d\xi \frac{1}{n\,(\xi)} \Bigg\{ \left[e^h \left(\frac{de^{-h}}{d\xi} \right) \right]_b^{\ a} \left[e^h \left(\frac{de^{-h}}{d\xi} \right) \right]_a^{T\ b} + \left[e^h \left(\frac{de^{-h}}{d\xi} \right) \right]_b^{\ a} \times \\ &\qquad \times \left[e^h \left(\frac{de^{-h}}{d\xi} \right) \right]_a^{\ b} - \left(\left[e^h \left(\frac{de^{-h}}{d\xi} \right) \right]_a^{\ a} \right)^2 + \sum_{u=1}^{q} \left(\frac{d\phi^u}{d\xi} \right)^2 \Bigg\} , \end{aligned} \qquad (2.22)$$

where the summation is performed over Lorentz indices. Note that the lower indices of e^{-h} and the upper indices of e^h cannot be lowered or raised by the Lorentz metric. To avoid confusion we label these indices with greek letters, namely we define 'vielbein'

$$e_\mu{}^a = (e^{-h})_\mu{}^a \qquad e_b{}^\mu = (e^h)_b{}^\mu\,; \qquad g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}\,. \tag{2.23}$$

Although we have not yet introduced a space-time, we shall name the a indices flat and the μ indices curved. As a result of the temporal involution and of the scalar product $\langle\ \rangle$ in $\mathcal{G}^{+++}$, the flat-index tensors have been endowed with a Lorentz metric while curved-index tensors define a metric in $\mathrm{GL}(D)/\,\mathrm{SO}(D-1,1)$. Hence, for any ξ, we are allowed to identify $g_{\mu\nu}(\xi)$ in eq. (2.23) as the metric tensor in S, eq. (2.1), at a fixed space-time point.

Using eq. (2.23), one can rewrite the action eq. (2.22) with flat or curved indices as

$$\mathcal{S}^{(0)} = \frac{1}{2}\int d\xi \frac{1}{n(\xi)}\left[e_b{}^\mu \frac{de_\mu{}^a}{d\xi}\left(\frac{de_{a\nu}}{d\xi}e^{\nu b} + e_a{}^\nu \frac{de_\nu{}^b}{d\xi}\right) - \left(e_b{}^\mu \frac{de_\mu{}^b}{d\xi}\right)^2 + \sum_{u=1}^{q}\left(\frac{d\phi^u}{d\xi}\right)^2\right] \tag{2.24}$$

or

$$\mathcal{S}^{(0)} = \frac{1}{2}\int d\xi \frac{1}{n(\xi)}\left[\frac{1}{2}\left(g^{\mu\nu}g^{\sigma\tau} - \frac{1}{2}g^{\mu\sigma}g^{\nu\tau}\right)\frac{dg_{\mu\sigma}}{d\xi}\frac{dg_{\nu\tau}}{d\xi} + \sum_{u=1}^{q}\frac{d\phi^u}{d\xi}\frac{d\phi^u}{d\xi}\right]. \tag{2.25}$$

At higher levels, the tensors multiplying the step operators couple nonlinearly to the level zero objects and between themselves. The coupling to the metric and to ϕ^u can be formally written down for all levels, but the self-coupling of the $A_{b_1\ldots b_s}{}^{a_1\ldots a_r}$ depend specifically on the group $\mathcal{G}$.

Consider a general A_{D-1} tensor $A_{a_1\ldots a_r}{}^{b_1\ldots b_s}$ parametrising a normalised step operator $R_{b_1\ldots b_s}{}^{a_1\ldots a_r}$. The commutation relations of $R_{b_1\ldots b_s}{}^{a_1\ldots a_r}$ with the $K^b{}_a$ are given by the tensor transformations as in eq. (2.4) and (2.6) and those with R_u have the form

$$[R_u, R_{b_1\ldots b_s}{}^{a_1\ldots a_r}] = \lambda_u\, R_{b_1\ldots b_s}{}^{a_1\ldots a_r}\,. \tag{2.26}$$

Here $\lambda_u = \sum \lambda_{u,i}$ where the $\lambda_{u,i}$ are the scale parameters of the simple step operators entering the multiple commutators defining $R_{b_1\ldots b_s}{}^{a_1\ldots a_r}$. This property follows from the Jacobi identity. Identifying for simple step operators $\lambda_{u,i}$ with $-\varepsilon a_I^u/2$ in eq. (2.8) we may identify for any ξ the $\phi^u(\xi)$ in $\mathcal{S}$, eq. (2.18), with the dilatons fields in S, eq. (2.1), at a fixed space time point. The particular $A_{a_1\ldots a_r}{}^{b_1\ldots b_s}(\xi)$ multiplying the step operators belonging to the subgroup $\mathcal{G}$ can be similarly identified to the corresponding potential forms in S along with their duals.

It is straightforward to compute the contribution $v^{(A)}$ to v of a given tensor when commutators of the $R_{b_1\ldots b_s}{}^{a_1\ldots a_r}$ between themselves are disregarded. On gets

$$v^{(A)} = \frac{1}{r!s!}dA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}\,\exp\left(-\sum_{u=1}^{q}\lambda^u\phi^u\right)\, e_{a_1}{}^{\mu_1}\cdots e_{a_r}{}^{\mu_r} e_{\nu_1}{}^{b_1}\cdots e_{\nu_s}{}^{b_s}\, R_{b_1\ldots b_s}{}^{a_1\ldots a_r}\,. \tag{2.27}$$

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The contribution $\mathcal{S}_0^{(A)}$ of $v^{(A)}$ to the action $\mathcal{S}$ is computed as previously and one gets

$$\mathcal{S}_0^{(A)}=\frac{1}{2}\int d\xi\frac{1}{n(\xi)}\left[\frac{1}{r!s!}\exp\left(-\sum_{u=1}^{q}2\lambda^u\phi^u\right)\frac{dA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}}{d\xi}g^{\mu_1\mu_1'}\cdots g^{\mu_r\mu_r'}g_{\nu_1\nu_1'}\cdots g_{\nu_s\nu_s'}\frac{dA_{\mu_1'\cdots\mu_r'}{}^{\nu_1'\cdots\nu_s'}}{d\xi}\right]. \tag{2.28}$$

The full action can only be approached in a recursive way. In $\mathcal{S}_0^{(A)}$, one must replace derivatives by non linear generalisations to take into account the non vanishing commutators between tensor step operators. We represent such terms by 'covariant' derivatives symbol $D/D\xi$. There evaluation is group dependent. As an example we give the covariant derivative of the 6-form appearing at level 2 in E_8^{+++}. One has, as in E_8^{++} [12],

$$\frac{D}{D\xi}A_{a_1a_2a_3a_4a_5a_6}=\frac{d}{d\xi}A_{a_1a_2a_3a_4a_5a_6}+10A_{[a_1a_2a_3}\frac{d}{d\xi}A_{a_4a_5a_6]}\,. \tag{2.29}$$

Formally the full action $\mathcal{S}$ is

$$\mathcal{S}\,=\mathcal{S}^{(0)}+\sum_A\mathcal{S}^{(A)}\,, \tag{2.30}$$

$$\begin{aligned}\mathcal{S}^{(A)}\,=\frac{1}{2}\int d\xi\frac{1}{n(\xi)}\bigg[\frac{1}{r!s!}\exp\left(-\sum_{u=1}^{q}2\lambda^u\phi^u\right)\times\\ \times\frac{DA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}}{d\xi}g^{\mu_1\mu_1'}\cdots g^{\mu_r\mu_r'}g_{\nu_1\nu_1'}\cdots g_{\nu_s\nu_s'}\frac{DA_{\mu_1'\ldots\mu_r'}{}^{\nu_1'\ldots\nu_s'}}{d\xi}\bigg],\end{aligned} \tag{2.31}$$

where the sum on A is a summation over all tensors appearing at all positive levels in the decomposition of $\mathcal{G}^{+++}$ into irreducible representations of A_{D-1}.

One may expand $\mathcal{S}$ given in eq. (2.30) in power of fields parametrising the positive step operators. Up to quadratic terms, the result $\mathcal{S}^{(Q)}$ is obtained by retaining in $v(\xi)$ terms independent or linear in these fields. Define the one-forms $[dA]$ and the moduli $p^{(a)}$ (or $p^{(\mu)}$), as in [8, 17], by[7]

$$\exp\left(-\sum_{u=1}^{q}\lambda^u\phi^u\right)\,\hat{e}_{a_1}{}^{\mu_1}\ldots\hat{e}_{a_r}{}^{\mu_r}\hat{e}_{\nu_1}{}^{b_1}\ldots\hat{e}_{\nu_s}{}^{b_s}\,\frac{dA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}}{d\xi}\stackrel{\text{def}}{=}\frac{[dA]_{a_1\ldots a_r}{}^{b_1\ldots b_s}}{d\xi}\,, \tag{2.32}$$

$$\hat{e}_b{}^{\mu}\frac{d\hat{e}_\mu{}^a}{d\xi}\stackrel{\text{def}}{=}\frac{dp^{(a)}}{d\xi}\,, \tag{2.33}$$

where $\hat{e}$ means that only the diagonal vielbein are kept. Taking into account that in eq. (2.24), only the first term in the right hand side contains non diagonal elements of the vielbein, we get

$$\begin{aligned}\mathcal{S}^{(Q)}\,=\int d\xi\frac{1}{n(\xi)}\bigg[\sum_{a=1}^{D}\left(\frac{dp^{(a)}}{d\xi}\right)^2-\frac{1}{2}\left(\sum_{a=1}^{D}\frac{dp^{(a)}}{d\xi}\right)^2+\frac{1}{2}\sum_{u=1}^{q}\left(\frac{d\phi^u}{d\xi}\right)^2+\\ +\frac{1}{2}\left(e_b{}^{\mu}\frac{de_\mu{}^a}{d\xi}\frac{de_{a\nu}}{d\xi}e^{\nu b}\right)^{(1)}+\frac{1}{2}\frac{1}{r!s!}\sum_A\frac{[dA]_{a_1\ldots a_r}{}^{b_1\ldots b_s}}{d\xi}\frac{[dA]_{b_1\ldots b_s}{}^{a_1\ldots a_r}}{d\xi}\bigg],\end{aligned} \tag{2.34}$$

[7]We can indifferently label the moduli by a curved or a flat index, as it is uniquely defined by the diagonal vielbein in the triangular gauge once the $g_{\mu\nu}$ have been chosen. The position of this index as a subscript or superscript is then a matter of convention and has no tensor significance.

where the superscript (1) in the vielbein term indicates that only terms quadratic in ${h_b}^a$ $(a > b)$ are kept. In the next section we shall produce solutions of the action eq. (2.34) which are *exact* solutions of the full action eq. (2.30).

3. BPS states in $\mathcal{G}^{+++}$

3.1 Extremal brane solutions of the oxidised actions S

We first recall how closed extremal branes are obtained as solutions of general relativity coupled to forms in the oxidised theories described by the actions eq. (2.1). For simplicity we restrict ourselves to one dilaton. One considers the diagonal metrics

$$ds^2 = -e^{2p^{(1)}} d\tau^2 + \sum_{\mu=2}^{D-p} e^{2p^{(\mu)}} (dx^\mu)^2 + \sum_{\lambda=1}^{p} e^{2p^{(D-p+\lambda)}} (dy^\lambda)^2 \,, \tag{3.1}$$

where y^λ label the p compact coordinates. The functions $p^{(\alpha)}$ $(\alpha = 1, 2, \dots D)$ depend only on the transverse coordinates x^μ in the non-compact dimensions and allow for multi-centre solutions. We choose $p \geq q_A$ where q_A is the dimension of the brane. If $q_A < p$, we take a lattice of q_A-branes in the compact directions transverse to the brane and average over them. Here q_A designates either an electrically charged q_E-brane with respect to the form p_I-form field strength F_{p_I}, or its magnetic dual q_M-brane.

One choose the ansätze for electric and magnetic branes

$$\text{Electric}: A_{\tau\lambda_1\dots\lambda_{q_E}} = \epsilon_{\tau\lambda_1\dots\lambda_{q_E}} E_E(\{x^\nu\}) \,, \tag{3.2}$$
$$\text{Magnetic}: \widetilde{A}_{\tau\lambda_1\dots\lambda_{q_M}} = \epsilon_{\tau\lambda_1\dots\lambda_{q_M}} E_M(\{x^\nu\}) \,, \tag{3.3}$$

where $\widetilde{A}$ is the (magnetic) potential of the dual field strength $\widetilde{F}$ defined by

$$\sqrt{-g} e^{a_I \phi} F^{\mu_1\dots\mu_{p_I}} = \frac{1}{(D-p_I)!} \epsilon^{\mu_1\dots\mu_{p_I}\,\nu_1\dots\nu_{D-p_I}} \widetilde{F}_{\nu_1\dots\nu_{D-p_I}} \,. \tag{3.4}$$

Extremal brane solutions of the coupled Einstein's equations, dilaton and forms are given by [26]

$$\begin{aligned} p_A^{(1)} &= -\frac{D-q_A-3}{\Delta} \ln H_A \,, \\ p_A^{(\mu)} &= \frac{q_A+1}{\Delta} \ln H_A \,, \\ p_A^{(D-p+\lambda)} &= \frac{\delta_A^{(\lambda)}}{\Delta} \ln H_A \,, \\ \phi_A &= \frac{D-2}{\Delta} \varepsilon_A \, a \ln H_A \,, \end{aligned} \tag{3.5}$$

where $H_A(\{x^\nu\})$ is a harmonic function related to the E_A by $H_A = \sqrt{2(D-2)/\Delta}\, E_A^{-1}$ and

$$\Delta = (q_A+1)(D-q_A-3) + \frac{1}{2} a_A^2 (D-2) \tag{3.6}$$

is invariant under electric-magnetic duality because $q_M = D - q_E - 4$. In eq. (3.5) $\delta_A^{(\lambda)} = -(D - q_A - 3)$ or $(q_A + 1)$ depending on wether y^λ is parallel or perpendicular to the q_A-brane. The factor ε_A is $+1$ for an electric brane and -1 for a magnetic one. The harmonic functions $H_A(\{x^\nu\})$ allow for parallel branes and are given (for $D - p > 3$) by

$$H_A = 1 + \sum_k \frac{Q_k}{|x^a - x_k^a|^{D-p-3}}\,, \tag{3.7}$$

where the x_k^a label the positions in non-compact space-time of the branes with charge Q_k.

The solutions eq. (3.5) satisfy [17]

$$(p + 3 - D)\, p_A^{(\mu)} = p_A^{(1)} + \sum_{\lambda=1}^{p} p_A^{(D-p+\lambda)} \qquad \mu = 2, \ldots, D - p\,, \tag{3.8}$$

and, using eq. (3.8), one verifies the differential relation

$$\sum_{\alpha=1}^{D} \left(dp_A^{(\alpha)}\right)^2 - \frac{1}{2}\left(\sum_{\alpha=1}^{D} dp_A^{(\alpha)}\right)^2 + \frac{1}{2}(d\phi_A)^2 = \frac{D-2}{\Delta}(d\ln H_A)^2\,. \tag{3.9}$$

We will now show how the result eqs. (3.9) characterising extremal branes, that is BPS configurations, are related for any $\mathcal{G}$ to exact solutions of the $\mathcal{G}^{+++}$ nonlinear action of section 2. The group properties of these solutions, as well as their interpretation as dualities in a very general context encompassing the familiar properties of string and M-theory, will emerge from these exact solutions.

3.2 BPS states as exact solutions of the $\mathcal{G}^{+++}$ actions $\mathcal{S}$

We shall look for solutions of the equations of motion derived from $\mathcal{S}$ and containing only one $A(\xi)$-field, or one non-diagonal $h(\xi)$-field, with *given indices*. For such solutions, we may disregard all non-linearity in the step operators. We shall prove this statement for the A-fields and hence also for the non-diagonal $h(\xi)$-field. The latter solutions will indeed follow from the former by Weyl transformations. First consider the non-linear terms arising from arising the 'covariant' D-derivatives in eq. (2.30). We label X the particular A-field component considered. We note that all terms in $DA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}$ are products of fields such that the sum of the levels of the factors in each term is equal to the level of $A_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}$. From the ordering of factors in the group element eq. (2.16), we see that all the fields contained in the 'covariant' derivatives are characterised by a level greater than zero. Consequently potentially dangerous terms containing more than one X-factor can not be present in the covariant derivative of X. For such term to appear in any covariant derivative it must contain a factor not equal to X. We can thus satisfy safely all the equations of motion of the fields of level greater than zero by putting to zero all A-fields different from X. The only term contributing to the equation of motion for X comes then from dX. We still have to consider the equations of motion of the level zero fields, namely we have to check that the $h_b{}^a$ with $b < a$ can be put consistently to zero when only one X-field is non-zero. This is the case as in eq. (2.28) the term quadratic in dX couples to metrics, which are either expressible in terms of diagonal vielbeins only or are at least quadratic in the non-diagonal ${h_b}^a$. Hence it is consistent to look for solutions of the equations of motion of $A(\xi)$ by replacing the action $\mathcal{S}$ by its simplified version eq. (2.34).

We shall as in eqs. (3.2) and (3.3), consider A to be an antisymmetric tensor with a time index τ and r space indices coupled to a step operator of the $\mathcal{G}$ subalgebra. The equation of motions are

a) The lapse constraint.

Eq. (2.34), taking eqs. (2.32) and (2.33) into account, reads

$$\sum_{\alpha=1}^{D}\left(\frac{dp^{(\alpha)}}{d\xi}\right)^2-\frac{1}{2}\left(\sum_{\alpha=1}^{D}\frac{dp^{(\alpha)}}{d\xi}\right)^2+\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2- \\ -\frac{1}{2}\exp\left[\varepsilon a\phi-2p^{(\tau)}-2\sum_{\lambda=\lambda_1}^{\lambda_r}p^{(\lambda)}\right]\left(\frac{dA_{\tau\lambda_1\ldots\lambda_r}}{d\xi}\right)^2=0\,. \tag{3.10}$$

Here we have taken one dilaton with scaling $\lambda=-\varepsilon a/2$ in accordance with eq. (2.8). Note that this relation is valid wether or not the magnetic root is simple, as seen in dimensional reduction. A crucial feature of this equation is the minus sign in front of the exponential. Its origin can be traced back to the temporal involution defining our coset space, hence to Lorentz invariance, because *both* magnetic and electric potentials have a time index.

b) The equation of motion for A.

We take the lapse $n(\xi)=1$. One gets

$$\frac{d}{d\xi}\left(\exp\left[\varepsilon a\phi-2p^{(\tau)}-2\sum_{\lambda=\lambda_1}^{\lambda_r}p^{(\lambda)}\right]\frac{dA_{\tau\lambda_1\ldots\lambda_r}}{d\xi}\right)=0\,. \tag{3.11}$$

c) The dilaton equation of motion.

$$-\frac{d^2\phi}{d\xi^2}-\frac{1}{2}\varepsilon a\exp\left[\varepsilon a\phi-2p^{(\tau)}-2\sum_{\lambda=\lambda_1}^{\lambda_r}p^{(\lambda)}\right]\left(\frac{dA_{\tau\lambda_1\ldots\lambda_r}}{d\xi}\right)^2=0\,. \tag{3.12}$$

d) The vielbein equations of motion.

$$-2\frac{d^2p^{(\alpha)}}{d\xi^2}+\sum_{\beta=1}^{D}\frac{d^2p^{(\beta)}}{d\xi^2}=0 \qquad \alpha\neq\tau\,,\lambda_i \qquad (i=1,2\ldots r)\,, \tag{3.13}$$

$$-2\frac{d^2p^{(\alpha)}}{d\xi^2}+\sum_{\beta=1}^{D}\frac{d^2p^{(\beta)}}{d\xi^2}+\exp\left[\varepsilon a\phi-2p^{(\tau)}-2\sum_{\lambda=\lambda_1}^{\lambda_r}p^{(\lambda)}\right]\times \\ \times\left(\frac{dA_{\tau\lambda_1\ldots\lambda_r}}{d\xi}\right)^2=0 \quad \alpha=\tau,\lambda_i\,. \tag{3.14}$$

We take as anzätze the solutions of the extremal brane problem but with H_A an unknown function of $H(\xi)$. Namely we pose

$$A_{\tau\lambda_1\ldots\lambda_r} = \epsilon_{\tau\lambda_1\ldots\lambda_r}\left[\frac{2(D-2)}{\Delta}\right]^{1/2} H^{-1}(\xi)\,, \tag{3.15}$$

$$p^{(\tau)} = p^{(\lambda_i)} = -\frac{D-r-3}{\Delta}\ln H(\xi)\,; \qquad p^{(\alpha)} = \frac{r+1}{\Delta}\ln H(\xi) \quad \alpha \neq \tau, \lambda_i\,, \tag{3.16}$$

$$\phi = \frac{D-2}{\Delta}\varepsilon a \ln H(\xi)\,. \tag{3.17}$$

From these equations and from eq. (3.6) we see that $\varepsilon a\phi - 2p^{(\tau)} - 2\sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)} = 2\ln H(\xi)$. It then follows that the equation of motion for A, eq. (3.11), reduces to, using eq. (3.15),

$$\frac{d^2 H(\xi)}{d\xi^2} = 0\,. \tag{3.18}$$

Given this result, it is straightforward to verify that the anzätze eqs. (3.15)–(3.17) satisfy the dilaton and the vielbein equations of motions. The lapse constraint takes the form

$$\sum_{\alpha=1}^{D}\left(dp^{(\alpha)}\right)^2 - \frac{1}{2}\left(\sum_{\alpha=1}^{D} dp^{(\alpha)}\right)^2 + \frac{1}{2}(d\phi)^2 - \frac{D-2}{\Delta}(d\ln H)^2 = 0\,, \tag{3.19}$$

where the differentials are taken in ξ-space. It has therefore exactly the same form in ξ-space as eq. (3.9) has in space-time. The relations eqs. (3.15), (3.16) and (3.17), together with eqs. (3.18) and the lapse constraint eq. (3.19) fully describe an exact solution of the full $\mathcal{G}^{+++}$ invariant action $\mathcal{S}$ defined recursively by eq. (2.30). We now discuss the significance of this result.

The eqs. (3.15)–(3.17) characterise completely the algebraic structure of the extremal brane solution but do not yield its harmonic character in space-time. As the functions $A_{\tau\lambda_1\ldots\lambda_r}(\xi)$, $p^{(\tau)}(\xi)$, $p^{(\lambda_i)}(\xi)$ and $\phi(\xi)$ were interpreted in the action $\mathcal{S}$ as functions at a fixed space-time point of the independent variable ξ, this is a consistent result. The solution $H = a + b\xi$ of eq. (3.18) would then describe a motion in the space of solutions, for instance of branes with different charges. However the fact that we have exact solutions of the action $\mathcal{S}$ with the correct algebraic structure of the extremal branes, means that these solutions are only indirectly related to the corresponding space-time solution. One expects that the information contained in this solution, which is of course not contained in a trivial constant space-time solution of the Einstein equation, is the required information to build coupled equations to higher space-time derivatives encoded in higher level representations, which would then be directly related to space-time solutions.

An indication that this is indeed the case is found by exploring the decomposition of the adjoint representation of E_8^{+++} into representations of A_{10}. There the fields at level 1,2,3 are the 3-form potential at level 1, the 6-form magnetic potential at level 2 and the dual graviton at level 3 [12, 15]. The corresponding step operators are R^{abc}, R^{abcdef} and the dual gravitons $R^{abcdefgh,k}$. They belong to the representations[8]

$$(0,0,1,0,0,0,0,0,0,0,0) \qquad (0,0,0,0,0,1,0,0,0,0,0) \qquad (1,0,0,0,0,0,0,0,1,0,0)\,. \tag{3.20}$$

Arbitrarily high number k of field derivatives could tentatively be described by the group parameters (or ξ-fields) of the representations

$$(0,0,1,0,0,0,0,0,0,0,k) \qquad (0,0,0,0,0,1,0,0,0,0,k) \qquad (1,0,0,0,0,0,0,0,1,0,k)\,. \tag{3.21}$$

One may verify that these indeed occur respectively at levels $7k+1, 7k+2, 7k+3$, although we have not proven, for $k > 2$, that their outer multiplicity does not vanish. This is however unlikely to happen as the corresponding roots are imaginary with squared length diverging as $k \to \infty$, and one expect outer multiplicities, which are already large[9] for $k = 2$ to grow rapidly with k. These towers are to some extent analogous to the towers which appear in E_8^{++} modulo 3 [12], but the representations modulo 7 of E_8^{+++} may be highly degenerate. Hence the identification would require a selection rule to isolate the relevant representations.

Such a selection rule can be imposed by taking, for instance, the *non degenerate* (i.e. of outer multiplicity one) representation

$$P_a = (0,0,0,0,0,0,0,0,0,0,1) \tag{3.22}$$

which appears at level 7. The negative root $-\alpha = \lambda_1$ of E_8^{+++} corresponding to a highest weight P^1 in A_{10} is given by (see Table 2 of [28])

$$\alpha = \alpha_1 + 3\alpha_2 + 5\alpha_3 + 7\alpha_4 + 9\alpha_5 + 11\alpha_6 + 13\alpha_7 + 15\alpha_8 + 10\alpha_9 + 5\alpha_{10} + 7\alpha_{11}\,. \tag{3.23}$$

Labelling by $\alpha_k^1, \alpha_k^2, \alpha_k^3$ the roots of E_{11} corresponding to the highest weights of the representations eq. (3.21), one verifies the additivity relations given in the last column of Table II. These show that the generators of the representations eq. (3.21) can be obtained by multicommutators[10] of P_a with R^{abc}, R^{abcdef} and $R^{abcdefgh,k}$. Thus the generators of the towers selected in this way could be identified with the 'derivative' representations, and P_a may be related to a generator of space-time translations. One then expects the functions $H(\xi)$ entering the above BPS solutions to be promoted to space-time harmonic functions when coupling with higher levels are taken into account, at least for E_8^{+++} solutions. From the generality of the correspondence between space-time and ξ-space appearing in the exact solutions, we may hope for a related mechanism to be operative in all $\mathcal{G}^{+++}$, although it might be in general not identical to the one suggested here. Indeed, the existence of an operator of the type displayed in eq. (3.22) cannot be present[11] in all $\mathcal{G}^{+++}$. Note also that there appears to be no direct relation between P_a, which is a *generator* of $\mathcal{G}^{+++}$, and the tentative identification of a momentum operator with a weight *not* contained in the adjoint representation of $\mathcal{G}^{+++}$ [30], as the latter would seem to imply an

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Dynkin indices	level	E_{11} root
$(0,0,0,0,0,0,0,0,0,0,1)$	7	α
$(0,0,1,0,0,0,0,0,0,0,k)$	$7k+1$	$\alpha^1_k = k\alpha + \alpha^1_0$
$(0,0,0,0,0,1,0,0,0,k)$	$7k+2$	$\alpha^2_k = k\alpha + \alpha^2_0$
$(1,0,0,0,0,0,0,1,0,k)$	$7k+3$	$\alpha^3_k = k\alpha + \alpha^3_0$

Table 2: Infinite towers of 'derivative' representations.

extension of the symmetry group beyond $\mathcal{G}^{+++}$. A more detailed account of these results and of their implications for the generation of space-time is differed to a separate publication [18].

3.3 Group theory of extremal branes, KK-waves and monopoles

The action $\mathcal{S}$ is invariant under non-linear, field dependent, $\mathcal{G}^{+++}$ transformations. Hence one can generate new solutions of $\mathcal{S}$ from the extremal brane solution. In this section we shall obtain solutions which can be put in correspondence with solutions of eq. (2.1), thereby testing the validity of $\mathcal{G}^{+++}$ in this restricted domain and putting into evidence the group-theoretical structure of these solutions.

We first note that the left hand side of the quadratic form eq. (3.9), or rather the corresponding quantity in ξ-space, eq. (3.19), occurs at level zero in the full $\mathcal{G}^{+++}$ invariant action $\mathcal{S}$. It appears in the first line of eq. (2.34), is equal to the bilinear form of $\mathcal{G}^{+++}$ restricted to its Cartan subalgebra, and is therefore invariant under the Weyl group of $\mathcal{G}^{+++}$ [8]. The appearance of a right-hand side in eq. (3.9) has, as will now be shown, a non trivial group-theoretical significance related to generators, which do not belong to this Cartan subalgebra but emerge in the non-linear realisation of $\mathcal{G}^{+++}$.

The solution eqs. (3.15)–(3.17) satisfy, in ξ-space, the relation eq. (3.8). This relation define an embedding of a subgroup $\mathcal{G}^{p+1}$ of $\mathcal{G}^{+++}$ acting on the p compact space dimensions in which the branes live and on the time dimension [17]. We shall consider the subgroup $\mathcal{G}^{p}$ of $\mathcal{G}^{p+1}$ which acts on the space dimensions only and we take $p \leq D-4$ so that $\mathcal{G}^{p+1}$ is a Lie group. This group is conjugate by a Weyl reflection in $\mathcal{G}^{+++}$ of the group $\mathcal{G}'^{\,p+1}$

[8]Here we follow the usual convention. The Dynkin labels of the A_{10} representations are labelled from right to left when compared with the labelling of the Dynkin diagram of Fig.1. For instance the last label on the right refers to the fundamental weight associated with the 'time' root labelled 1 in figure 1.

[9]For k=2, the outer multiplicities for the three towers are respectively 3290, 8567 and 36067 ! [27].

[10]We have however been unable to disprove the accidental vanishing of some relevant commutator.

[11]This was pointed out to us by Axel Kleinschmidt. The periodicity argument based on the observation that representations at any level l occur in the direct product of representations of level one implies for E^{+++} that representations with Dynkin labels eq. (3.22) may occur at level l when $10 = 3l \ mod\ 11$ (or dual ones when $1 = 3l \ mod\ 11$). Indeed one does get such representations at level 7, 18, ... (4, 15, ...). (The representations at level 4 and 7 have outer multiplicity one while the outer multiplicities at level 18 and 15 are respectively 15765 and 824 [27].) For some $\mathcal{G}^{+++}$, in particular for D_{24}^{+++}, there are no such solutions to the periodicity constraint [29].

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obtained by deleting the first $D-p-1$ nodes of the gravity line [17] and hence $\mathcal{G}^p$ is conjugate to its subgroup $\mathcal{G}'^p$ characterising the usual dimensional reduction of eq. (2.1) to $D-p$ dimensions.

We shall consider the transformations mapping *one* root to another root, thereby generating solutions of the same 'family' as the extremal solution just described. These transformations include the Weyl group $W(\mathcal{G}^{+++})$ of $\mathcal{G}^{+++}$. We shall examine Weyl transforms of the extremal brane solution characterised by one positive step operator which send the positive root into a positive root. Such transformations leave invariant not only $\mathcal{S}$ but also preserves their quadratic truncation eq. (2.34). Hence eq. (3.19) is invariant under the Weyl group of $\mathcal{G}^{p+1}$. The restriction to the Weyl group of $\mathcal{G}^p$ selects transformed fields with one time index.

Thus $W(\mathcal{G}^p)$ leaves invariant the quadratic form

$$\sum_{a=1}^{D}(dp^{(a)})^2+\frac{1}{2}\Bigg[-\left(\sum_{i=1}^{D}dp^{(a)}\right)^2+\sum_{u=1}^{q}(d\phi^u)^2+\left(e_b{}^{\mu}de_{\mu}{}^{a}de_{a\nu}e^{\nu b}\right)^{(1)}+ \\ +\frac{1}{r!s!}\sum_{A}[dA]_{a_1\ldots a_r}^{\;b_1\ldots b_s}[dA]_{b_1\ldots b_s}^{\;a_1\ldots a_r}\Bigg], \tag{3.24}$$

and the embedding relation eq. (3.8) in ξ-space. It acts on A-fields, or non-diagonal vielbeins, containing one time index. The sum of the first three terms is the invariant metric of $\mathcal{G}^{+++}$ restricted to its Cartan subgroup. Together with the embedding relation eq. (3.8) they are left invariant under the Weyl group of $\mathcal{G}^{p+1}$. The relevance of this invariance for extremal branes (for all simply laced $\mathcal{G}^{+++}$ algebras) and for intersecting brane configurations was pointed out in reference [17]. However no group-theoretical interpretation was provided for the appearance of the last term in the relation eq. (3.19), or in the right hand side of eq. (3.9). We see here that it arises from the Weyl transformations of the step operators and the additional terms in eq. (3.24) guarantee the invariance of this relation under $W(\mathcal{G}^p)$. They allow, for all $\mathcal{G}^{+++}$, the generation by Weyl transformations of new solutions from one extremal brane solution. We stress again that in the present approach both electric and magnetic branes are described 'electrically'.

In appendix C, it is shown how well-known duality symmetries of M-theory are interpreted in the present context, including the duality transformations of branes, KK waves and KK monopoles (Taub-NUT spaces). Such transformations are however not a privilege of M-theory and occur in *all* $\mathcal{G}^{+++}$ invariant actions. This is exemplified below, taking for definiteness the action $\mathcal{S}$ for the group E_7^{+++} which is related to the action S eq. (A.37) of gravity coupled to a 4- and a 2- form field strength in 9 space-time dimensions. The Dynkin diagram of E_7^{+++} is depicted in Fig.1, which exhibits the two simple electric roots (10) and (9) corresponding respectively to the step operators $R^{7\,8\,9}$ and R^9 which couple to the electric potentials $A_{7\,8\,9}$ and A_9 (see appendix A).

We take as input the electric extremal 2-brane $\mathbf{e}_{(8,9)}$ in the directions $(8,9)$ associated with the 4-form field strength whose corresponding potential is $A_{1\,8\,9}$ and submit it to the non trivial Weyl reflection W_{10} associated with the electric root (10) of figure 1. We display below, both for $\mathbf{e}_{(8,9)}$ and its transform, the moduli, i.e. the vielbein components $p^{(a)}$ and the the dilaton value ϕ, of the brane solution eqs. (3.16) and (3.17) as a ten-dimensional vector where the last component is the dilaton. We also indicate the transform of the step operator $R^{1\,8\,9}$ under the Weyl transformation.[12] The transformed vector follows from eq. (A.45). We obtain

$$\left(-4,3,3,3,3,3,3,-4,-4;2\sqrt{7}\right)\frac{\ln H(\xi)}{14} \qquad \mathbf{e}_{(8,9)} \qquad R^{1\,8\,9} \tag{3.25}$$
$$\downarrow W_{10}$$
$$(-7,0,0,0,0,0,7,0,0;0)\frac{\ln H(\xi)}{14} \qquad \mathbf{kk}_{\mathbf{e}\,(7)} \qquad K^{1}{}_{7}\,. \tag{3.26}$$

The transformed brane is, according to the analysis of appendix B, a KK-wave in the direction 7, in analogy with the double T-duality in M-theory acting on the generators eq. (C.6).

We now move the electric brane through Weyl reflections associated with roots of the gravity line to $\mathbf{e}_{(5,9)}$ and submit it to the Weyl reflection W_{10}. We now find that the brane $\mathbf{e}_{(5,9)}$ is invariant but moving it to the position $\mathbf{e}_{(5,6)}$, we get

$$\left(-4,3,3,3,-4,-4,3,3,3;2\sqrt{7}\right)\frac{\ln H(\xi)}{14} \qquad \mathbf{e}_{(5,6)} \qquad R^{1\,5\,6} \tag{3.27}$$
$$\downarrow W_{10}$$
$$\left(-1,6,6,6,-1,-1,-1,-1,-1;4\sqrt{7}\right)\frac{\ln H(\xi)}{14} \qquad \mathbf{m}_{(5,6,7,8,9)} \qquad R^{1\,5\,6\,7\,8\,9}\,. \tag{3.28}$$

This is a magnetic 5-brane in the directions $(5,6,7,8,9)$ associated to the 2-form field strength ! It is expressed in terms of its dual potential $A_{1\,5\,6\,7\,8\,9}$. Submit instead $\mathbf{e}_{(5,9)}$ to to the Weyl reflection W_9 associated with the electric root (9) of figure 1. The transformed vector is now deduced from eq. (A.44). The 2-brane $\mathbf{e}_{(5,9)}$ is again invariant, but moving it to to the position $\mathbf{e}_{(5,6)}$, we now get

$$\left(-4,3,3,3,-4,-4,3,3,3;2\sqrt{7}\right)\frac{\ln H(\xi)}{14} \qquad \mathbf{e}_{(5,6)} \qquad R^{1\,5\,6}$$
$$\downarrow W_{9}$$
$$\left(-3,4,4,4,-3,-3,4,4,-3;-2\sqrt{7}\right)\frac{\ln H(\xi)}{14} \qquad \mathbf{m}_{(5,6,9)} \qquad R^{1\,5\,6\,9} \tag{3.29}$$

This is a magnetic 3-brane in the directions $(5,6,9)$ associated to the 4-form field strength, expressed in terms of its dual potential $A_{1\,5\,6\,9}$. These properties are reminiscent of the double T-duality in M-theory acting on the generators eq. (C.8) and (C.9), but are more involved, due to the interplay of the 2- and 4- form field strength in the action eq. (A.37).

Finally, let us submit the magnetic 5-brane $\mathbf{m}_{(5,6,7,8,9)}$ obtained in eq. (3.28) to the Weyl reflection W_9. One obtains

$$\left(-1,6,6,6,-1,-1,-1,-1,-1;4\sqrt{7}\right)\frac{\ln H(\xi)}{14} \qquad \mathbf{m}_{(5,6,7,8,9)} \qquad R^{1\,5\,6\,7\,8\,9}$$
$$\downarrow W_{9}$$
$$(0,7,7,7,0,0,0,0,-7;0)\frac{\ln H(\xi)}{14} \qquad \mathbf{kk}_{\mathbf{m}\,(2,3,4;9)} \qquad R^{1\,5\,6\,7\,8\,9,9} \tag{3.30}$$

[12]In appendix C the transformations of the step operators have been explicitly computed for E_8^{+++}.

eq. (3.30) describes, as in M-theory under the transformation eq. (C.11), a purely gravitational configuration, namely a KK-monopole (see appendix B eq. (B.12)) with transverse directions (2,3,4) and Taub-NUT direction (9) in terms of a dual gravity tensor $h_{1\,5\,6\,7\,8\,9,\,9}$.

It is also possible, as in appendix C, to generate solutions not contained, at least explicitly, in the group $\mathcal{G}^p$. These are very interesting solutions as they may test the significance of genuine Kac-Moody extensions of the Lie groups. Such analysis is outside the scope of the present paper where we test only solutions which can straightforwardly be mapped to space time solutions of the effective actions eq. (2.1).

The above example illustrate the analogy of M-theory duality transformations with similar transformations in all 'M-theories' defined by all $\mathcal{G}^{+++}$. One may indeed carry the same analysis for all $\mathcal{G}^{+++}$, using the results of appendix A, and exhibit for each of them the 'duality' transformations of the branes. As in M-theory, these dualities are symmetries in non-compact space-time. This is because $\mathcal{G}^{p+1}$ is, as $\mathcal{G}'^{\,p+1}$, the Lie group symmetry of the action eq. (2.1) dimensionally reduced to three dimensions (for $p = D-4$). They differ because while the latter reduction leaves a lorentzian non-compact space-time, the former leads to a Euclidean space-time by compactifying time. The group $\mathcal{G}^p$ of transformations on ξ-space discussed above, is thus in one to one correspondence with the group $\mathcal{G}^p$ of space-time transformations when time is decompactified. In particular, the functions $H(\xi)$ can thus be mapped into harmonic functions $H(\{x^\nu\})$. However as pointed out in the previous section, more work is needed to relate directly $H(\xi)$ to $H(\{x^\nu\})$, and solutions in ξ-space to solutions in space-time for all $\mathcal{G}^{+++}$, through translation operators hopefully induced by group generators, as suggested by eq. (3.22).

4. Conclusion

The basic concept underlying the present approach is the tentative inclusion of *local* symmetries in an infinite *global* symmetry. We find the results obtained in the present work indicative of the possibility of such an embedding whose consequences for consistent theories of gravity and matter may be far reaching. They would not require an explicit implementation of diffeomorphism and gauge invariance which should emerge dynamically and might lead to the inclusions of many new degrees of freedom.

We have constructed in a recursive way invariant actions for all global symmetries $\mathcal{G}^{+++}$. They are not defined in space-time but there are definite indications that the latter may emerges from $\mathcal{G}^{+++}$. Further work is necessary to ascertain wether or not generators of E_8^{+++} can be related to space-time translations and if some encoding of space-time can be generalised for all $\mathcal{G}^{+++}$. Exact solutions have been presented, which contain all the algebraic structure of extremal BPS charged brane solutions and BPS gravitational branes, namely Kaluza-Klein waves and KK-monopoles (Taub-NUT spaces). The transformation properties of these solutions put into evidence the general group-theoretical origin of 'dualities' for all $\mathcal{G}^{+++}$, which apparently do not require an underlying string theory.

These transformations also allow for solutions, alluded to in section 3.3 and appendix C, which are not related in an obvious way to Einstein's gravity. Such solutions occur when a step operator associated with a parameter of low level, which does have an interpretation in terms of a field of the oxidised theory, is mapped under Weyl transformation on a step operator of higher level. The significance of such solutions remains to be unveiled. Further study should tell whether they fit into the framework of general relativity as we know it, or into some generalisation of it. Recall that the effective action of the bosonic string and of the bosonic sector of M-theory are maximally oxidised theories and hence can be extended to $\mathcal{G}^{+++}$ theories. One may hope that the degrees of freedom of string theories, possibly in the tensionless limit, have their counterpart in the $\mathcal{G}^{+++}$ theories. In such a perspective the present approach may constitute a first step towards a consistent theory of gravity which would encompass string theories and allow for different matter content.

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A. $\mathcal{G}^{+++}$ and Weyl reflexions

A.1 A_{D-3}^{+++}

This case corresponds to pure gravity. Its Dynkin diagram is displayed in figure 1. The rank of A_{D-3}^{+++} is $n = D$

The simple roots of A_{D-3}^{+++} are $E_m = K^m{}_{m+1}$, $m = 1, \ldots, n-1$ and $E_n = R^{4\ldots n,n}$ where $R^{a_1\ldots a_{n-3},b}$ is totally anti-symmetric in its a indices and the part antisymmetrised in all its indices vanishes [7, 15]. This generator of A_{D-3}^{+++} transforms under $SL(D)$ according to eq. (2.6),

$$[K^a{}_b, R^{c_1\ldots c_{n-3},d}] = \delta^{c_1}_b R^{a\ldots c_{n-3},d} + \cdots + \delta^d_b R^{c_1\ldots c_{n-3},a} \,. \tag{A.1}$$

The cartan generators in the Chevalley basis are given by [8]

$$\begin{aligned} H_m &= (K^m{}_m - K^{m+1}{}_{m+1}) \qquad a = 1, \dots, n-1 \\ H_n &= -(K^1{}_1 + K^2{}_2 + K^3{}_3) + K^n{}_n \,. \end{aligned} \tag{A.2}$$

The non-trivial Weyl reflexion W_n, namely the one which does not correspond to a simple root in the gravity line but to the simple root α_n has already be discussed in [8]. This Weyl reflexion induces the following changes on the parameters

$$\begin{aligned} p'^m &= p^m + (p^4 + \cdots + p^{n-1}) + 2p^n \qquad m = 1, 2, 3 \\ p'^m &= p^m \qquad m = 4, \dots, n-1 \\ p'^n &= -p^n - (p^4 + \cdots + p^{n-1}) \,. \end{aligned} \tag{A.3}$$

A.2 B^{+++}_{D-2}

This non simply laced case has already been sketched in [8]. Its Dynkin diagram is displayed in figure 1. The maximally oxidised theory associated with B^{+++}_{D-2} is characterised by the following action

$$S = \int d^D x \sqrt{-g^{(D)}} \left(R^{(D)} - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2.3!} e^{-l\phi} F_{\mu\nu\sigma} F^{\mu\nu\sigma} - \frac{1}{2.2!} e^{-(l/2)\phi} F_{\mu\nu} F^{\mu\nu} + \text{C.S.} \right), \tag{A.4}$$

where l is given by

$$l = \left(\frac{8}{D-2} \right)^{1/2} . \tag{A.5}$$

The rank n of B^{+++}_{D-2} is $n = D + 1$. Thus one must add to the GL(D) generator $K^a{}_b$ a commuting generator R associated with the dilaton. In addition to the simple step operators $K^m{}_{m+1}$ one has two other step operators R^D and $R^{5\dots D}$ associated to the two simple roots, one short and one long, which do not belong to the gravity line. These generators belongs to the A_{D-1} representations R^a and $R^{a_1 \dots a_{n-5}}$ that are respectively a 1 and $n-5$ rank anti-symmetric tensors. From eq. (2.8) they satisfy the following equations

$$[R, R^a] = \frac{l}{4} R^a \,, \qquad [R, R^{a_1 \dots a_{n-5}}] = -\frac{l}{2} R^{a_1 \dots a_{n-5}} \quad . \tag{A.6}$$

The cartan generators of B^{+++}_{D-2} in the Chevalley basis are given by

$$H_m = (K^m{}_m - K^{m+1}{}_{m+1}) \qquad m = 1, \dots, n-2 \tag{A.7}$$

$$H_{n-1} = -\frac{2}{(D-2)}(K^1{}_1 + \ldots + K^{n-2}{}_{n-2}) + \frac{2(D-3)}{(D-2)} K^{n-1}{}_{n-1} + l\, R \tag{A.8}$$

$$H_n = -\frac{(D-4)}{(D-2)}(K^1{}_1 + \ldots + K^4{}_4) + \frac{2}{(D-2)}(K^5{}_5 + \ldots K^{n-1}{}_{n-1}) - l\, R \,. \tag{A.9}$$

There are two non-trivial Weyl reflexions W_{n-1} and W_n corresponding respectively to the simple roots α_{n-1} and α_n.

W_{n-1}. The Weyl reflexion induces the following changes on the generators

$$\begin{aligned} K^m{}_m &\to K^m{}_m \qquad a = 1, \dots, n-2 \\ K^{n-1}{}_{n-1} &\to K^{n-1}{}_{n-1} - H_{n-1} \\ R &\to R - \frac{2}{l(D-2)} H_{n-1}\,, \end{aligned} \tag{A.10}$$

which yield the parameter transformations

$$\begin{aligned} p'^m &= p^m + \frac{2}{(D-2)} p^{n-1} + \frac{l}{2\,(D-2)}\phi \qquad m = 1, \dots n-2 \\ p'^{n-1} &= \frac{(4-D)}{(D-2)} p^{n-1} - \frac{l\,(D-3)}{2\,(D-2)}\phi \\ \phi' &= \frac{(D-4)}{(D-2)}\phi - l\, p^{n-1}\,. \end{aligned} \tag{A.11}$$

W_n. The Weyl reflexion induces the following changes on the generators

$$\begin{aligned} K^m{}_m &\to K^m{}_m \qquad m = 1, \dots, 4 \\ K^m{}_m &\to K^m{}_m - H_n \qquad m = 5, \dots, n-1 \\ R &\to R + \frac{l}{2} H_n\,, \end{aligned} \tag{A.12}$$

which yield the parameter transformations

$$\begin{aligned} p'^m &= p^m + \frac{D-4}{D-2}\left(p^5 + \cdots + p^{n-1}\right) - \frac{l\,(D-4)}{2\,(D-2)}\phi \qquad m = 1, \dots, 4 \\ p'^m &= p^m - \frac{2}{D-2}(p^5 + \cdots + p^{n-1}) + \frac{l}{(D-2)}\phi \qquad m = 5, \dots, n-1 \\ \phi' &= \phi + l\,(p^5 + \ldots + p^{n-1}) - \frac{4}{(D-2)}\phi\,. \end{aligned} \tag{A.13}$$

A.3 C_{q+1}^{+++}

The Dynkin diagram characterising this non-simply laced theory is given in figure 1. The corresponding maximally oxidised theory is defined in four dimensions and is characterised by q different dilatons. The action is [4]

$$S = \int d^4x\, \sqrt{-g^{(4)}}\left(R^{(4)} - \frac{1}{2}\sum_{u=1}^{q} \partial_\mu \phi^u \partial^\mu \phi^u - \frac{1}{2}\sum_{\alpha} e^{\sigma_\alpha\cdot\phi} F^\alpha_\mu F^{\alpha\mu} - \frac{1}{2.2!}\sum_{i=1}^{q} e^{e_i\cdot\phi} F^i_{\mu\nu} F^{i\mu\nu}\right), \tag{A.14}$$

where $\sigma_\alpha = \{2e_i, e_j \pm e_i \text{ with } j > i\}$ and the e_i are q-dimensional vectors $(0, \dots, 1, \dots 0)$ with the 1 at the i^{th} position. For each dilaton ϕ^u with $u = 1 \dots q$ one introduces a generator R_u commuting with the $K^a{}_b$.

The simple roots which do not belong to the gravity line are of two kinds. First there are the electric roots of the one-form field strengths F_1^α with α such that $\sigma_\alpha = e_{i+1} - e_i$ with $i = 1 \dots q-1$ and $\sigma_\alpha = 2e_1$. We associate with these roots respectively $q-1$ generators denoted S_j with $j = 1, \dots, q-1$ and a generator S_q. They are in a scalar representation of A_3. Second there is the magnetic root associated to the two-form field strength F_2^q with dilaton coupling e_q. We associate with this simple root a step operator R^4 which belong to a vectorial representation R^a of A_3.

To summarise we have the following simple step operators for C_{q+1}^{+++}

$$\begin{aligned} E_m &= K^m{}_{m+1} \qquad m = 1, 2, 3 \\ E_4 &= R^4\,, \qquad E_{4+j} = S_j \qquad j = 1, \dots, q\,. \end{aligned} \tag{A.15}$$

We deduce from the lagrangian eq. (A.14), knowing the electric or magnetic nature of the simple roots and eq. (2.8), the following relations for $u = 1, \dots, q$

$$\begin{aligned} [R_u, R^a] &= \frac{1}{2}\delta_{q,u}\, R^a \\ [R_u, S_j] &= -\frac{1}{2}\left(\delta_{u,q+1-j} - \delta_{u,q-j}\right) S_j \qquad j = 1, \dots, q-1 \\ [R_u, S_q] &= -\delta_{u,1}\, S_q\,. \end{aligned} \tag{A.16}$$

From the commutators eq. (A.16) we can determine the Cartan generators in the Chevalley basis in terms of those the K-basis. We get

$$H_m = (K^m{}_m - K^{m+1}{}_{m+1}) \qquad m = 1, \dots, 3 \tag{A.17}$$

$$H_4 = -(K^1{}_1 + K^2{}_2 + K^3{}_3) + K^4{}_4 + 2\,R_q \tag{A.18}$$

$$H_{4+m} = -2\,R_{q+1-m} + 2\,R_{q-m} \qquad m = 1, \dots, q-1 \tag{A.19}$$

$$H_{4+q} = -2\,R_1\,. \tag{A.20}$$

We are now in position to describe the non-trivial Weyl reflexions

W_4. The Weyl reflexion corresponding to the simple root α_4 induces the following changes on the generators

$$\begin{aligned} K^m{}_m &\to K^m{}_m \qquad a = 1, \dots, 3 \\ K^4{}_4 &\to K^4{}_4 - H_4 \\ R_q &\to R_q - \frac{1}{2}H_4\,, \end{aligned} \tag{A.21}$$

which yield the parameter transformations

$$\begin{aligned} p'^m &= p^m + p^4 + \frac{1}{2}\phi^q \qquad m = 1, \dots, 3 \\ p'^4 &= -\frac{1}{2}\phi^q \\ \phi'^q &= -2p^4 \\ \phi'^u &= \phi^u \qquad u = 1, \dots q-1\,. \end{aligned} \tag{A.22}$$

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W_{4+j}, ($j = 1, \ldots, q-1$). Using eqs. (A.17)–(A.20) one gets the Weyl reflexions W_{4+j} for $j = 1, \ldots q-1$. They induce the following transformations on the parameters

$$\begin{aligned} p'^m &= p^m \qquad m = 1, \ldots, 4 \\ \phi'^{q+1-j} &= \phi^{q-j}\,, \qquad \phi'^{q-j} = \phi^{q+1-j}\,. \end{aligned} \tag{A.23}$$

The others ϕ^u are left invariant. These transformations interchange two neighbouring dilatons.

W_{4+q}. Using eqs. (A.17)–(A.20) one gets the Weyl reflexions W_{4+q}. They induce the following transformation of the parameters

$$\begin{aligned} p'^m &= p^m \qquad && m = 1, \ldots, 4 \\ \phi'^u &= \phi^u && u = 2, \ldots, q \\ \phi'^1 &= -\phi^1\,. \end{aligned} \tag{A.24}$$

This is an S-duality like transformation on the first dilaton.

A.4 D_{D-2}^{+++}

The rank n is $n = D + 1$. There is thus a dilaton and its associated generator R. The Dynkin diagram characterising this theory is given in Fig.1. The corresponding maximally oxidised theory is defined in D dimensions and for $D = 26$ it is the low-energy effective action of the bosonic string (without the tachyon). The action is

$$S = \int d^D x\, \sqrt{-g^D}\left(R^D - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.3!}e^{-l\phi}F_{\mu\nu\sigma}F^{\mu\nu\sigma}\right), \tag{A.25}$$

with l given by eq. (A.5). D_{D-2}^{+++} has already been discussed in the literature [6, 16, 8] and in particular the non-trivial Weyl reflexions have been presented in [8]. We list here the results for sake of completness.

The cartan generators are given by [6] (see [8] for the normalisation used here)

$$\begin{aligned} H_m &= K^m{}_m - K^{m+1}{}_{m+1} \qquad m = 1, \ldots, n-2 \\ H_{n-1} &= -\frac{2}{(D-2)}(K^1{}_1 + \ldots + K^{n-3}{}_{n-3})\frac{(D-4)}{(D-2)}(K^{n-2}{}_{n-2} + K^{n-1}{}_{n-1}) + l\,R \\ H_n &= -\frac{(D-4)}{(D-2)}(K^1{}_1 + \ldots + K^4{}_4) + \frac{2}{(D-2)}(K^5{}_5 + \ldots + K^{n-1}{}_{n-1}) - l\,R\,. \end{aligned} \tag{A.26}$$

There are two non-trivial Weyl reflexion W_{n-1} and W_n corresponding respectively to the simple roots α_{n-1} and α_n. They induce the parameter transformations

W_{n-1}.

$$\begin{aligned} p'^m &= p^m + \frac{2}{(D-2)}(p^{n-2} + p^{n-1}) + \frac{l}{(D-2)}\phi \qquad m = 1, \ldots, n-3 \\ p'^m &= p^m - \frac{(D-4)}{(D-2)}(p^{n-2} + p^{n-1}) - \frac{l\,(D-4)}{2\,(D-2)}\phi \qquad m = n-2, n-1 \\ \phi' &= \frac{(D-6)}{(D-2)}\phi - l(p^{n-2} + p^{n-1})\,. \end{aligned} \tag{A.27}$$

W_n.

$$
\begin{aligned}
p'^m &= p^m + \frac{D-4}{D-2}\left(p^5 + \ldots + p^{n-1}\right) - \frac{l\,(D-4)}{2\,(D-2)}\phi \qquad m = 1, \ldots, 4 \\
p'^m &= p^m - \frac{2}{D-2}\left(p^5 + \ldots + p^{n-1}\right) + \frac{l}{(D-2)}\phi \qquad m = 5, \ldots, n-1 \\
\phi' &= \frac{(D-6)}{(D-2)}\phi + l\,\left(p^5 + \ldots + p^{n-1}\right). \qquad \text{(A.28)}
\end{aligned}
$$

A.5 E_6^{+++}

The Dynkin diagram characterising this theory is given in figure 1. The corresponding maximally oxidised theory is defined in eight dimensions and is characterised by one dilaton, a four-form and a one-form field strength. The action is given by [4]

$$
S = \int d^8x \sqrt{-g^{(8)}}\left(R^{(8)} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.4!}e^{-\phi}F_{\mu\nu\sigma\rho}F^{\mu\nu\sigma\rho} - \frac{1}{2}e^{2\phi}F_\mu F^\mu + \text{C.S.}\right). \quad \text{(A.29)}
$$

Once again there is a generator R associated to the dilaton and commuting with the $K^a{}_b$. In addition to the simple step operators of the gravity line one has two other step operators $R^{6\,7\,8}$ and S corresponding respectively to the electric simple root of the four-form field strength and to electric simple root of the one-form field strength. The $R^{6\,7\,8}$ generator belongs to the A_7 representation which is a third rank antisymmetric tensor R^{abc} obeying thus eq. (2.7). The remaining generator S is a A_7-scalar. Considering the dilaton couplings in eq. (A.29) and using eq. (2.8), one gets the following commutators

$$
[R, R^{abc}] = \frac{1}{2}R^{abc}, \qquad [R, S] = -S. \qquad \text{(A.30)}
$$

Using eq. (A.30) one can then express the Cartan generator in terms of the (K, R)-basis. One gets

$$
\begin{aligned}
H_m &= \left(K^m{}_m - K^{m+1}{}_{m+1}\right) \qquad m = 1, \ldots, 7 \qquad &\text{(A.31)} \\
H_8 &= -\frac{1}{2}\left(K^1{}_1 + \ldots + K^5{}_5\right) + \frac{1}{2}(K^6{}_6 + K^7{}_7 + K^8{}_8) + R \qquad &\text{(A.32)} \\
H_9 &= -2\,R. \qquad &\text{(A.33)}
\end{aligned}
$$

There are two non-trivial Weyl reflexions W_8 and W_9 corresponding respectively to the simple roots α_8 and α_9.

W_8. The Weyl reflexion induces the following changes on the generators

$$
\begin{aligned}
K^m{}_m &\to K^m{}_m \qquad a = 1, \ldots, 5 \\
K^m{}_m &\to K^m{}_m - H_8 \qquad a = 6 \ldots 8 \\
R &\to R - \frac{1}{2}H_8, \qquad \text{(A.34)}
\end{aligned}
$$

which yield the parameter transformations

$$
\begin{aligned}
p'^m &= p^m + \frac{1}{2}\left(p^6+p^7+p^8\right) + \frac{1}{4}\phi \qquad m=1,\ldots,5 \\
p'^m &= p^m - \frac{1}{2}\left(p^6+p^7+p^8\right) - \frac{1}{4}\phi \qquad m=6,\ldots,8 \\
\phi' &= \frac{1}{2}\phi - \left(p^6+p^7+p^8\right).
\end{aligned} \tag{A.35}
$$

W_9. This Weyl reflexion gives

$$
\begin{aligned}
p'^m &= p^m \qquad m=1,\ldots,8 \\
\phi' &= -\phi ,
\end{aligned} \tag{A.36}
$$

which is an S-duality like transformation.

A.6 E_7^{+++}

The Dynkin diagram characterising this theory is given in Fig.1. The corresponding maximally oxidised theory is defined in nine dimensions and is characterised by one dilaton, a four-form field strength and a two-form field strength. The action is given by [4]

$$
S = \int d^9x\,\sqrt{-g^{(9)}}\left(R^{(9)} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.4!}e^{l\phi}F_{\mu\nu\sigma\rho}F^{\mu\nu\sigma\rho} - \frac{1}{2.2!}e^{-2l\phi}F_{\mu\nu}F^{\mu\nu} + \text{C.S.}\right), \tag{A.37}
$$

with

$$
l = \frac{2}{\sqrt{7}}. \tag{A.38}
$$

One has a generator R associated with the dilaton and commuting with the $K^a{}_b$. In addition to the simple step operators of the gravity line one has two other step operators $R^{7\,8\,9}$ and R^9 corresponding respectively to the electric simple root of the four-form field strength and to electric simple root of the two-form field strength. The $R^{7\,8\,9}$ generator belongs to the A_8 representation which is a third rank antisymmetric tensor R^{abc} and the remaining generator R^9 belongs to a A_8-vector R^a. Considering the dilaton couplings in eq. (A.37) and using eq. (2.8), one gets the following commutators

$$
[R, R^{abc}] = -\frac{l}{2}\,R^{abc}, \qquad [R, R^a] = l\,R^a . \tag{A.39}
$$

Using eq. (A.39) one can then express the Cartan generator in terms of the (K, R)-basis. One gets

$$
H_m = \left(K^m{}_m - K^{m+1}{}_{m+1}\right) \qquad m=1,\ldots,8 \tag{A.40}
$$

$$
H_9 = -\frac{1}{7}\left(K^1{}_1+\ldots+K^8{}_8\right) + \frac{6}{7}K^9{}_9 + 2l\,R \tag{A.41}
$$

$$
H_{10} = -\frac{3}{7}(K^1{}_1+\ldots+K^6{}_6) + \frac{4}{7}(K^7{}_7+K^8{}_8+K^9{}_9) - l\,R . \tag{A.42}
$$

There are two non-trivial Weyl reflexions W_9 and W_{10} corresponding respectively to the simple roots α_9 and α_{10}.

W_9. The Weyl reflexion induces the following changes on the generators

$$\begin{aligned} K^m{}_m &\to K^m{}_m \qquad a = 1, \ldots, 8 \\ K^9{}_9 &\to K^9{}_9 - H_9 \\ R &\to R - l\, H_9 \,, \end{aligned} \tag{A.43}$$

which yield the parameter transformations

$$\begin{aligned} p'^m &= p^m + \frac{1}{7}p^9 + \frac{l}{7}\phi \qquad m = 1, \ldots, 8 \\ p'^9 &= \frac{1}{7}p^9 - \frac{6l}{7}\phi \\ \phi' &= -\frac{1}{7}\phi - 2l\, p^9 \,. \end{aligned} \tag{A.44}$$

W_{10}. This Weyl reflexion gives

$$\begin{aligned} p'^m &= p^m + \frac{3}{7}\left(p^7 + p^8 + p^9 - \frac{l}{2}\phi\right) \qquad m = 1, \ldots, 6 \\ p'^m &= p^m - \frac{4}{7}\left(p^7 + p^8 + p^9 - \frac{l}{2}\phi\right) \qquad m = 7, \ldots, 9 \\ \phi' &= \frac{5}{7}\phi + l\,\left(p^7 + p^8 + p^9\right) . \end{aligned} \tag{A.45}$$

A.7 E_8^{+++}

The Dynkin diagram characterising this theory is given in figure 1. The corresponding maximally oxidised theory is the bosonic sector of eleven dimensional supergravity whose action is

$$S = \int d^{11}x \sqrt{-g^{(11)}} \left(R^{(11)} - \frac{1}{2.4!}F_{\mu\nu\sigma\tau}F^{\mu\nu\sigma\tau} + CS\right) . \tag{A.46}$$

E_8^{+++} has already been extensively studied in the literature [6, 15, 8] and in particular the non-trivial Weyl reflexions have been presented in [8]. Again we list them here for sake of completness.

The cartan generator are given by [6]

$$\begin{aligned} H_m &= \left(K^m{}_m - K^{m+1}{}_{m+1}\right) \qquad m = 1, \ldots, 10 \\ H_{11} &= -\frac{1}{3}\left(K^1{}_1 + \ldots + K^8{}_8\right) + \frac{2}{3}\left(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11}\right) . \end{aligned} \tag{A.47}$$

The non-trivial Weyl transformation W_{11} corresponding to the simple root α_{11} is

$$\begin{aligned} K'^a{}_a &= K^a{}_a \qquad a = 1, \ldots, 8 \\ K'^a{}_a &= K^a{}_a - H_{11} \qquad a = 9, 10, 11 \,. \end{aligned} \tag{A.48}$$

yielding

$$\begin{aligned} p'^a &= p^a + \frac{1}{3}(p^9 + p^{10} + p^{11}) \qquad a = 1, \ldots, 8 \\ p'^a &= p^a - \frac{2}{3}(p^9 + p^{10} + p^{11}) \qquad a = 9, 10, 11 \,. \end{aligned} \tag{A.49}$$

A.8 F_4^{+++}

The Dynkin diagram characterising this non simply laced theory is given in figure 1. The corresponding maximally oxidised theory is defined in six dimensions and is characterised by one dilaton, a one-form field strength, two two-form field strengths and two three-form field strengths. The action is given by [4]

$$S = \int d^6x\sqrt{-g^{(6)}}\Big(R^{(6)} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.3!}e^{-l\phi}F^{(1)}_{\mu\nu\sigma}F^{(1)\mu\nu\sigma} - \frac{1}{2.3!}F^{(2)}_{\mu\nu\sigma}F^{(2)\mu\nu\sigma} - \\ - \frac{1}{4}e^{-l/2\phi}F^{(1)}_{\mu\nu}F^{(1)\mu\nu} - \frac{1}{4}e^{\frac{l}{2}\phi}F^{(2)}_{\mu\nu}F^{(2)\mu\nu} - \frac{1}{2}e^{l\phi}F_\mu F^\mu + \text{C.S.}\Big) . \quad \text{(A.50)}$$

with

$$l = \sqrt{2} . \quad \text{(A.51)}$$

One has a generator R associated with the dilaton and commuting with the $K^a{}_b$. In addition to the simple step operators of the gravity line one has two other step operators R^6 and S corresponding respectively to the electric simple root of the two-form field strength $F_2^{(1)}$ and to electric simple root of the one-form field strength. The R^6 generator belongs to the vectorial A_5 representation and the remaining generator S is a A_5 scalar. Considering the dilaton couplings in eq. (A.50) and using eq. (2.8), one gets the following commutators

$$[R, R^a] = +\frac{l}{4}R^a , \qquad [R, S] = -\frac{l}{2}S . \quad \text{(A.52)}$$

Using eq. (A.52) one can express the Cartan generator in terms of the (K, R)-basis. One gets

$$H_m = (K^m{}_m - K^{m+1}{}_{m+1}) \qquad m = 1, \dots, 5 \quad \text{(A.53)}$$

$$H_6 = -\frac{1}{2}(K^1{}_1 + \cdots + K^5{}_5) + \frac{3}{2}K^6{}_6 + \frac{2}{l}R \quad \text{(A.54)}$$

$$H_7 = -\frac{4}{l}R . \quad \text{(A.55)}$$

There are two non-trivial Weyl reflexions W_6 and W_7 corresponding respectively to the simple roots α_6 and α_7.

W_6. The Weyl reflexion induces the following changes on the generators

$$\begin{aligned} K^m{}_m &\to K^m{}_m \qquad a = 1, \dots, 5 \\ K^6{}_6 &\to K^6{}_6 - H_6 \\ R &\to R - \frac{l}{4}H_6 , \end{aligned} \quad \text{(A.56)}$$

which yield the parameter transformations

$$\begin{aligned} p'^m &= p^m + \frac{1}{2}p^6 + \frac{l}{8}\phi \qquad m = 1, \dots, 5 \\ p'^6 &= -\frac{1}{2}p^6 - \frac{3l}{8}\phi \\ \phi' &= \frac{1}{2}\phi - \frac{2}{l}p^6 . \end{aligned} \quad \text{(A.57)}$$

W_7. This Weyl reflexion gives

$$\begin{aligned} p'^m &= p^m \qquad m = 1, \ldots, 6 \\ \phi' &= -\phi \,, \end{aligned} \tag{A.58}$$

which is an S-duality like transformation.

A.9 G_2^{+++}

The Dynkin diagram characterising this non simply laced theory is given in figure 1. The corresponding maximally oxidised theory is defined in five dimensions with no dilatons and a two-form field strength. The action is given by [4]

$$S = \int d^5x \sqrt{-g^{(5)}} \left(R^{(5)} - \frac{1}{2.2!} F_{\mu\nu} F^{\mu\nu} \right) + \text{C.S.} \,. \tag{A.59}$$

In addition to the simple step operators of the gravity line one has another step operator R^5 corresponding to the electric simple root of the two-form. The generator R^5 belongs to a A_4-vector R^a. The Cartan generators are given by

$$H_m = (K^m{}_m - K^{m+1}{}_{m+1}) \qquad m = 1, \ldots, 4 \tag{A.60}$$

$$H_5 = -(K^1{}_1 + \ldots + K^4{}_4) + 2K^5{}_5 \,. \tag{A.61}$$

The non-trivial Weyl reflexion W_5 corresponding to the simple roots α_5 yields the transformations

$$\begin{aligned} p'^m &= p^m + p^5 \qquad m = 1, \ldots, 4 \\ p'^5 &= -p^5 \,. \end{aligned} \tag{A.62}$$

B. Description of the KK solutions

B.1 The KK-momentum solution

We derive here the group parameters describing the KK-momentum solution in D dimensions. It is a purely gravitational uncharged solution.

The KK momentum solution in, say, the direction D is given by the metric

$$ds^2 = -H^{-1}(dx^1)^2 + \sum_{\mu=2}^{D-1} (dx^\mu)^2 + H\left[dx^D + (H^{-1} - 1)dx^1\right]^2 \,, \tag{B.1}$$

where $H(\{x^\mu\})$ with $\mu = 2 \ldots D-1$ is a harmonic function in $D-2$ dimensions. In the triangular gauge the vielbein are given by

$$\begin{aligned} e_1{}^1 &= H^{-1/2} \\ e_D{}^D &= H^{1/2} \\ e_1{}^D &= H^{-1/2} - H^{1/2} \,. \end{aligned} \tag{B.2}$$

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We now use eq. (2.23) to find the moduli ${h_a}^b$ describing the KK-momentum. For the diagonal components in the triangular gauge one has

$$e_i{}^i = \left(e^{-h}\right)_i{}^i = e^{-h_i{}^i}\,, \tag{B.3}$$

where $i = 1$ or D. We have thus the following non-zero moduli

$$h_1{}^1 = -p^1 = \frac{1}{2}\ln H, \qquad h_D{}^D = -p^D = -\frac{1}{2}\ln H\,, \tag{B.4}$$

and

$$h_1{}^1 = -h_D{}^D\,. \tag{B.5}$$

The remaining non-diagonal vielbein gives

$$\begin{aligned} e_1{}^D &= (e^{-h})_1{}^D \\ &= -h_1{}^D\left[1+\sum_{n=2}^{\infty}\frac{1}{n!}(-1)^{n-1}\left(\sum_{a=0}^{n-1}(h_1{}^1)^a\,(h_D{}^D)^{n-a-1}\right)\right]. \end{aligned} \tag{B.6}$$

Using eq. (B.5) in eq. (B.6) one gets

$$e_1{}^D = -\frac{h_1{}^D}{2h_1{}^1}\,(e^{h_1{}^1} - e^{-h_1{}^1})\,. \tag{B.7}$$

Combining eq. (B.7) with eqs. (B.2) and (B.4) we find

$$h_1{}^D = \ln H\,. \tag{B.8}$$

The KK-momentum is thus entirely described by the three non vanishing group parameters given in eqs. (B.4) and (B.8).

B.2 The KK-monopole solution

We will discuss here the group parameters describing the KK-monopole solution in D dimensions. The KK-monopole solution, in the longitudinal directions $(x^2, \ldots, x^{D-4})$ and Taub-NUT direction x^D is given by the metric

$$\begin{aligned} ds^2 = &-\left(dx^1\right)^2 + \left(dx^2\right)^2 + \cdots + \left(dx^{D-4}\right)^2 + H^{-1}\left(dx^D + \sum_{i=1}^{3} A_i^D dx^{D-4+i}\right)^2 + \\ &+ H\sum_{i=1}^{3} dx^{D-4+i}dx^{D-4+i}\,, \end{aligned} \tag{B.9}$$

where H a harmonic function in the 3 transverse dimensions $x^{D-3}, x^{D-2}, x^{D-1}$ and

$$F_{ij}{}^D \equiv \partial_i A_j^D - \partial_j A_i^D = -\epsilon_{ijk}\,\partial_k H \qquad i = D-3, D-2, D-1\,. \tag{B.10}$$

The only non-zero field of this solution which does not correspond to a generator of the Cartan subalgebra may be taken to be the potential $h_{1\,2\ldots D-4\,D,D}$ of a field dual to $F_{ij}{}^D$. This yields an 'electric' description of the KK-monopole.

To see this let us first consider the diagonal vielbein. From eq. (B.9) the non-trivial ones are given by

$$\begin{aligned} e_D{}^D &= H^{-1/2} \\ e_i{}^i &= H^{1/2} \qquad i = D-3, D-2, D-1\,, \end{aligned} \tag{B.11}$$

and the corresponding moduli $p^{(a)} = -h_a{}^a = \ln e_a{}^a$ are

$$\begin{aligned} p^1 &= 0 \\ p^i &= 0 \qquad i = 2\dots D-4 \\ p^i &= \frac{1}{2}\ln H \qquad i = D-3, D-2, D-1 \\ p^D &= -\frac{1}{2}\ln H\,. \end{aligned} \tag{B.12}$$

The non-diagonal vielbein are

$$e_i{}^D = H^{-1/2} A_i^D \quad \text{or} \quad A_i^D = e_i{}^D \left(e^{-1}\right)_D{}^D \qquad i = D-3, D-2, D-1\,. \tag{B.13}$$

One defines a field strength dual to $F_{ij}{}^D$

$$\sqrt{-g}\widetilde{F}^{1\dots D-4\ k\, D,D} = \frac{1}{2}\epsilon^{1\dots D-4\ k\, i\, j\, D} F_{ij}{}^D\,. \tag{B.14}$$

Using the Bianchi identity on the RHS and eq. (B.10) one gets the equation of motion for the H-field,

$$\partial_k \sqrt{-g}\widetilde{F}^{1\dots D-4\ k\, D,D} = \sum_i \partial_i\partial_i H = 0 \qquad i = D-3, D-2, D-1\,. \tag{B.15}$$

The vielbein matrix and its inverse, evaluated from eqs. (B.11) and (B.13) yields $g_{DD} = g^{ii} = H^{-1}$, $g^{ij} = 0 \quad i \neq j$ and $\sqrt{-g} = H$. Inserting these values in eq. (B.14) and using eq. (B.10) we see that the quantity $h_{1\,2\dots D-4\,D,D}$ given by

$$h_{1\,2\dots D-4\,D,D} = \epsilon_{1\,2\dots D-4\,D}\,(1/H) \tag{B.16}$$

is the potential of the dual field strength $\widetilde{F}_{1\,2\dots\,D-4\ k\,D,D}$.

C. Step operators and "dualities" in E_8^{+++}

In section 3.3, it has been proven that for all $\mathcal{G}^{+++}$, BPS solutions transform into each other under Weyl transformations of $\mathcal{G}^{+++}$. These transformation appear in the non-compact dimensions as duality symmetries. BPS solutions are characterised by only *one* field associated to *one* step operator in addition to the moduli associated to the Cartan generators. This is a consequence of the fact that all the BPS branes are always described as electrically charged in $\mathcal{G}^{+++}$. The transformations of BPS solutions under Weyl reflexions is thus entirely determined by the Weyl transformation of the step operators. In this appendix, we illustrate this fact in the familiar case of M-theory, i.e. $\mathcal{G}^{+++} = E_8^{+++}$ where the only non-trivial Weyl reflexion is associated with the electric root α_{11} (see Fig.1). This transformation has furthermore an interpretation in type IIA in terms of a double T-duality in the (9) and (10) directions followed by an exchange of the two directions. We shall thus compare the mapping of step operators by Weyl transformations with these duality transformations.

We recall that the step operators corresponding to the simple roots of E_8^{+++} are

$$E_m = K^m{}_{m+1} \quad a = 1, \dots, 10 \qquad \text{and} \qquad E_{11} = R^{9\,10\,11}\,. \tag{C.1}$$

They satisfy eq. (2.4) and eq. (2.7). Recall that the representations of A_{10} occuring up to level 3 are $(0,0,1,0,0,0,0,0,0,0)$, $(0,0,0,0,0,1,0,0,0,0)$ and $(1,0,0,0,0,0,0,0,1,0,0)$. The corresponding fields and generators are $h_b{}^a$ and $K^b{}_a$ at level 0, A_{abc} and R^{abc} at level 1, A_{abcdef} and R^{abcdef} at level 2 and finally $h_{a_1\dots a_8,b}$ and $R^{a_1\dots a_8,b}$ at level 3 which are totally antisymmetric in the a indices and the antisymmetrised part in all its indices vanishes [6, 12]. In the normalisation eq. (2.9) we have the following relation between the operators of level 1 and level 2

$$[R^{abc}, R^{def}] = R^{abcdef}\,. \tag{C.2}$$

The action of the electric Weyl reflexion W_{11} on the Cartan subalgebra is given in eq. (A.48). On the positive generators belonging to GL(D), one gets, using eq. (2.4) and eq. (2.7),

$$K'^i{}_j = K^i{}_j \qquad i < j \leq 8 \tag{C.3}$$
$$K'^9{}_{10} = K^9{}_{10} \qquad K'^{10}{}_{11} = K^{10}{}_{11} \tag{C.4}$$
$$K'^i{}_9 = R^{i\,10\,11} \qquad K'^i{}_{10} = R^{i\,11\,9} \quad i = 1 \dots 8 \tag{C.5}$$
$$K'^i{}_{11} = R^{i\,9\,10} \qquad i = 1 \dots 8\,. \tag{C.6}$$

These transformations are in agreement with the interpretation of W_{11} as a double T-duality plus echange in the (9) and (10) directions. Indeed in type IIA a non-zero $h_1{}^9$ or $h_1{}^{10}$ corresponds to a KK-momentum in the (9) or (10) direction (see appendix B). Performing the double duality plus the exchange give a $1F$ in the (10) or (9) direction. Uplifting back to M-theory this gives a $M2$-brane in (10 11) or in the directions (9 11) described by a non-vanishing $A_{1\,10\,11}$ or $A_{1\,9\,11}$ which is consistent with eq. (C.3) with $i = 1$. Similarly a non-zero $h_1{}^{11}$ describes in type IIA to a $D0$ brane which gives after the duality and uplifting a $M2$-brane in the (9 10) directions, in agreement with eq. (C.6).

The transformations of the remaining R^{abc} under W_{11} give

$$R'^{\,9\,10\,11} = R_{9\,10\,11} \tag{C.7}$$
$$R'^{ija} = -R^{ija} \qquad i, j \leq 8, \quad a > 8 \tag{C.8}$$
$$R'^{ijk} = R^{ijk\,9\,10\,11} \qquad i, j, k \leq 8\,. \tag{C.9}$$

The first relation simply states that under W_{11}, α_{11} is the only positive root mapped to a negative one. The second and third relations are consistent with the T-duality interpretation on the branes. Consider first eq. (C.8) with a time index. The non zero A_{1ia} describes an $M2$-brane in the (ia) directions. There are two cases to distinguish: $a = 11$ and $a = 9$ or 10. If $a = 11$ it corresponds in the IIA language to a $1F$ in the (i) direction which is indeed invariant under the double T-duality plus exchange. In the case $a = 9$ we have a $D2$-brane[13] in the $(i9)$ directions which upon the double T-duality in the $(9\,10)$ directions gives a $D2$ in $(i10)$, and the exchange of the (9) and (10) directions gives back the original $D2$. The uplifted $M2$ is thus invariant and eq. (C.8) is consistent with the duality picture on the $M2$ brane. Consider now eq. (C.9) with atime index. The $M2$ brane lying in the (ij) directions is described by a non zero A_{1ij}. This $M2$ yields a $D2$ in type IIA that under the double T-duality plus exchange is mapped onto a $D4$ brane in the direction $(ij\,9\,10)$. When uplifted to 11 dimensions it yields an $M5$ in the $(ij\,9\,10\,11)$ directions described by a non-zero $A_{1ij\,9\,10\,11}$, in agreement with eq. (C.9).

We finally analyse the transformations of the remaining R^{abcdef} under W_{11}. We get

$$R'^{ijklab} = R^{ijklab} \qquad i,j \leq 8\,, \quad a > 8 \tag{C.10}$$

$$R'^{ijklma} = R^{ijklmbca,a} \qquad i,j,k \leq 8 \quad a \neq b \neq c > 8\,. \tag{C.11}$$

Note first that we did not list the transformation of $R^{a_1\cdots a_6}$ with all the six indices $a_i \leq 8$. These yield operators of level 4. The associated parameters do not have an obvious interpretation in terms of eleven-dimensional supergravity fields and their interpretation in terms of branes has not yet been clarified. This is not too surprising because it corresponds in the usual U-duality discussion of M-theory to a case where the transverse non-compact space is of dimension two and where the significance of the U-duality orbits is unclear [21].

Consider first eq. (C.10). This equation implies that an $M5$ with two longitudinal directions $(9), (10)$ or (11)is invariant under the Weyl transformation. There are two cases. First, none of the two directions is (11). In type IIA one has then a $NS5$ longitudinal in $(9\,10)$, which is indeed separately invariant under the double T-duality and under the exchange. Second, one of the two directions is (11). In this case upon dimensional reduction one gets a $D4$ along one of the two $(9), (10)$ directions, which is also invariant under the conjugate action of the double T-duality and of the exchange of the (9) and (10) directions.

We now consider eq. (C.11). It is relevant for an $M5$ with only one longitudinal direction (a) along $(9), (10)$ or (11). The case $a = 11$ corresponds in type IIA to a $D4$. Performing the double T-duality plus exchange one gets a $D6$ which is uplifted in M-theory to a $KK6$ monopole with $a = 11$ as the Taub-NUT direction. The case $a = 9$ (the discussion is similar for $a = 10$) corresponds in type IIA to a $NS5$ brane with $a = 9$ as one of its longitudinal directions The T-duality in the longitudinal direction $a = 9$ leaves the $NS5$ invariant but the T-duality in the transverse (10) direction maps the $NS5$ onto a $KK5$ monopole with (10) as the Taub-NUT direction. Finally, the exchange $(9) \leftrightarrow (10)$ selects $a = 9$ as the Taub-NUT direction of the $KK5$. Thus the interpretation of the Weyl reflexion W_{11} as a duality is consistent with the electric description of the $KK6$-monopole of M-theory in terms of only *one* non-vanishing field of level 3, namely the field $h_{t\,y_1\ldots y_6\,b,b}$ associated to the step operator $R^{t\,y_1\cdots y_6\,b,b}$ with y_i as the longitudinal coordinates and b as the Taub-NUT direction. The KK-monopole being a purely gravitational solution, it

[13] The same argument can be repeated for $a = 10$.

does not depend on the possible presence of form field strengths and may, for all $\mathcal{G}^{+++}$, be similarly characterised by a field $h_{t\,y_1\ldots y_{D-5}\,b,b}$ with y_i as the longitudinal coordinates and b as the Taub-NUT direction.

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